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AN INVERSION FORMULA INVOLVING PARTITIONS

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In this note we outline a combinatorial proof of an inversion formula involving partitions of a number. This formula can be used to obtain the theory of symmetric group characters in a purely combinatorial way, as will be done in a forthcoming book, *The combinatorics of the symmetric group*, by the present author and Dr. G.-C. Rota.

The terminology we use is as follows. By a composition α of an integer n we mean a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of nonnegative integers whose sum is n. A partition of n is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > 0$. The notation $\lambda \vdash n$ means "λ is a partition of n". We use the symbols α, β for compositions, λ, μ, ρ for partitions.

A Young diagram of shape λ is an array of dots, with λ_1 dots in the first row, λ_2 in the second row, etc., in which the first dots from the rows lie in a column, the second dots form a column, and so on. The conjugate partition $\tilde{\lambda}$ of λ is the shape obtained when the Young diagram of shape λ is transposed about its main diagonal, i.e., the rows of the transposed diagram are the columns of the original diagram. A generalized Young tableau (GYT) π of shape λ is an array of integers q_{ij} $(i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, \lambda_i)$ with $q_{ij} > 0, q_{i,j+1} \geqq q_{ij}$ if $j < \lambda_i$, and $q_{i+1,j} > q_{ij}$ if $j \leqq \lambda_{i+1}$, i.e., an array of positive integers of shape λ which is increasing nonstrictly along the rows and increasing strictly down the columns. The type of a GYT π is the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ of n (where $\lambda \vdash n$), where α_i is the number of times the integer i appears in π.

If $\alpha = (\alpha_1, \ldots, \alpha_s)$ is a composition of n with $s \leqq n$, and $\tau \in S_n$ (the symmetric group on $\{1, 2, \ldots, n\}$), then $\tau \cdot \alpha$ is the composition of n whose parts are $\alpha_i + \tau(i) - i$, $i = 1, 2, \ldots, n$ (where $\alpha_i = 0$ if $i > s$), if all these parts are nonnegative, and $\tau \cdot \alpha$ is undefined otherwise. We also define $\tau * \lambda$ to be the partition of n whose parts are $\lambda_i + \tau(i) - i$ in nonincreasing order if all these parts are nonnegative, and $\tau * \lambda$ is undefined otherwise.

Our inversion formula can now be stated.

THEOREM. *Let f, g be mappings from $\{\lambda | \lambda \vdash n\}$ to some field F of characteristic* 0. *Then*

$$f(\lambda) = \sum_{\tau \in S_n} (\text{sign } \tau) g(\tau * \lambda) \leftrightarrow g(\lambda) = \sum_{\mu \vdash n} K_{\mu\lambda} f(\mu),$$

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where $K_{\mu\alpha}$ is the number of GYT π of shape μ and type α.

It should be noted that this formula follows immediately from the fact that $e_\lambda = \sum_\tau (\text{sign}\,\tau) h_{\tau*\lambda}$ and $h_\lambda = \sum_\mu K_{\mu\lambda} e_\mu$, where the h's and the e's are respectively the complete homogeneous symmetric functions and the Schur functions, since $\{h_\lambda | \lambda \vdash n\}$ and $\{e_\lambda | \lambda \vdash n\}$ are linearly independent sets. However, since one of our uses of the theorem is to prove just this connection between the h's and the e's, we want to prove the theorem from first principles. To do this we state a number of lemmas and outline some of their proofs.

Let $\leqq$ be the partial ordering on $\{\lambda \vdash n\}$ given by $\lambda \leqq \mu$ iff $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leqq \mu_1 + \mu_2 + \cdots + \mu_i \forall_i$.

LEMMA 1. *$K_{\lambda\mu} \neq 0$ implies $\lambda \geqq \mu$, and $K_{\lambda\lambda} = 1$.*

COROLLARY. *The matrix $(K_{\lambda\mu})$ is nonsingular (with determinant 1).*

Let $m(\alpha, \beta)$ be the number of matrices with entries 0 and 1 with the sum of the ith row being α_i, the sum of the jth column being β_j.

LEMMA 2. $m(\lambda, \mu) = \sum_\rho K_{\rho\lambda} K_{\tilde{\rho}\mu}$.

PROOF. Knuth's dual correspondence gives a constructive proof of this fact.

COROLLARY. *The matrix $(m(\lambda, \mu))$ is nonsingular (with determinant ± 1).*

PROOF. By Lemma 2, $(m(\lambda, \mu)) = (K_{\lambda\mu})^T \cdot (K_{\tilde{\lambda}\mu})$. $(K_{\lambda\mu})$ has determinant 1, and $(K_{\tilde{\lambda}\mu}) = P \cdot (K_{\lambda\mu})$, where P is the permutation matrix of the permutation $\lambda \to \tilde{\lambda}$ (i.e., $P = (p_{\lambda\mu})$, with $p_{\lambda\mu} = 1$ if $u = \tilde{\lambda}$, $= 0$ if $\mu \neq \tilde{\lambda}$).

LEMMA 3. $K_{\tilde{\lambda}\alpha} = \sum_{\tau\in S_n} (\text{sign}\,\tau) m(\tau \cdot \lambda, \alpha)$.

FIRST PROOF. We outline a proof as follows.

Let $M(\alpha, \beta)$ be the set of 0-1 matrices with row sums $(\alpha_1, \alpha_2, \ldots)$, column sums $(\beta_1, \beta_2, \ldots)$. To $A \in M(\tau \cdot \lambda, \alpha)$, let (t, i, j) be the least triple (ordered lexicographically) such that $i < j$ and $\rho_i^{(t)} - \tau(i) = \rho_j^{(t)} - \tau(j)$, where $\rho_i^{(t)}$ is the sum of the first t entries in the ith row of A, if any such triple exists. If (t, i, j) exists, switch the first t entries of the ith row with those of the jth and call the resulting matrix B. The following facts can be shown to hold for the correspondence $A \to B$.

(i) If $A \to B$ via the triple (t, i, j) and $A \in M(\tau \cdot \lambda, \alpha)$, then $B \in M((\tau \circ (ij)) \cdot \lambda, \alpha)$, so that the contributions of A and B in $\sum_{\tau\in S_n} (\text{sign}\,\tau) m(\tau \cdot \lambda, \alpha)$ cancel each other out.

(ii) If $A \to B$ then $B \to A$.

(iii) No triple (t, i, j) exists for $A \in M(\tau \cdot \lambda, \alpha)$ iff $\tau = 1 (=$ identity in $S_n)$ and $\rho_i^{(t)} - i > \rho_j^{(t)} - j$ for all triples (t, i, j) with $i < j$, or equivalently

$\tau = 1$ and $\rho_i^{(t)} \geqq \rho_j^{(t)} \forall (t, i, j)$ with $i < j$.

From (i), (ii), and (iii), it follows that $\sum_{\tau \in S_n} (\text{sign } \tau) m(\tau \cdot \lambda, \alpha)$ is the number of $A \in M(\lambda, \alpha)$ satisfying the condition in (iii). Now to each such $A \in M(\lambda, \alpha)$ associate the array π of shape $\tilde{\lambda}$ and type α letting the kth column of π consist of those j such that $a_{kj} = 1$ (where $A = (a_{ij})$), ordered in increasing fashion down the column. It is not difficult to show that π is a GYT, and that every GYT of shape $\tilde{\lambda}$ and type α arises in this way, proving the result.

SECOND PROOF. If a_k is the elementary symmetric function of degree k on variables $x_1, x_2, \ldots, x_n$ and if $\Delta = \det(x_i^{n-j})$, we can obtain the equality in Lemma 3 by computing $a_{\alpha_s} \cdot a_{\alpha_{s-1}} \cdot \cdots a_{\alpha_1} \cdot \Delta$ in two different ways.

COROLLARY. $K_{\tilde{\lambda}\alpha} = \sum_{\tau \in S_n} (\text{sign } \tau) m(\tau * \lambda, \alpha)$.

PROOF. $m(\alpha, \beta)$ does not depend on the order of the entries in the sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$, so $m(\tau * \lambda, \alpha) = m(\tau \cdot \lambda, \alpha)$.

It should be noted at this point that for the same reason as is given in the proof of the above corollary, it follows from Lemma 3 that $K_{\lambda\alpha}$ does not depend on the order of the entries in α.

Finally, we prove the theorem.

PROOF OF THEOREM. It is easily shown that it suffices to find nonsingular matrices $(a_{\lambda\mu})$, $(b_{\lambda\mu})$ such that $a_{\lambda\mu} = \sum_{\tau \in S_n} (\text{sign } \tau) b_{\tau*\lambda,\mu}$ and $b_{\lambda\mu} = \sum_\rho K_{\rho\lambda} a_{\rho\mu}$. But by what we have proved already, we can take $a_{\lambda\mu} = K_{\tilde{\lambda}\mu}$, $b_{\lambda\mu} = m(\lambda, \mu)$, so we are done.

Note. This inversion formula is similar to a Möbius inversion, for it is equivalent to the fact that the functions $\phi(\lambda, \mu) = K_{\mu\lambda}$ and $\psi(\lambda, \mu) = \sum_{\tau \in S_n \text{ s.t. } \tau*\lambda = \mu} (\text{sign } \tau)$ are inverse to each other in the incidence algebra of $\{\lambda | \lambda \vdash n\}$ with respect to the ordering $\leqq$.

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