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DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY SPHERES

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Introduction. The purpose of this note is to announce some results on free actions of S^1 and S^3 on homotopy spheres. In the following, most of the discussion of S^3 actions will be omitted since it is completely analogous to S^1 actions. Let S^1 act on $S^{2p-1} \times S^{2q-1}$ by $g(x, y) = (gx, gy)$ for $g \in S^1$ and $(x, y) \in S^{2p-1} \times S^{2q-1}$. It is always assumed that $q \leqq p$. This is a free action and let $K^{p+q,q}$ be the orbit space. Here is the motivation for this work. Let f be a diffeomorphism of $K^{p+q,q}$ which is homotopic to the identity. Let $\bar{f}$ be its covering which is an equivariant diffeomorphism of $S^{2p-1} \times S^{2q-1}$. The manifold $\Sigma(\bar{f}) = S^{2p-1} \times D^{2q} \cup_{\bar{f}} D^{2p} \times S^{2q-1}$ obtained by gluing along $S^{2p-1} \times S^{2q-1}$ via $\bar{f}$ is a homotopy sphere. $\Sigma(\bar{f})$ supports a free S^1 action defined by $g(x, y) = (gx, gy)$ where $g \in S^1$ and $(x, y) \in S^{2p-1} \times D^{2q}$ or $D^{2p} \times S^{2q-1}$. It is easy to check that this action depends only on the pseudo-isotopy class α of f and will be denoted by $(\Sigma(\alpha), S^1)$. Let $P(\alpha)$ be the orbit space. Note that $(\Sigma(\alpha), S^1)$ is a free S^1 action on homotopy $(2p + 2q - 1)$-sphere with standard characteristic $(2q - 1)$-sphere i.e. the induced action on which is linear. Let $A^{n,q}$ be the set of all free S^1 actions on homotopy $(2n - 1)$-spheres with standard characteristic $(2q - 1)$-spheres. For $q = [(n + 1)/2]$, $A^n = A^{n,q}$ is the set of all decomposable S^1 actions on homotopy $(2n - 1)$-spheres. Similarly let B^n be the set of all decomposable S^3 actions on homotopy $(4n - 1)$-spheres (see [**6**]). For $x \in A^{n,q}$, let $s_{2k}(x)$ be the splitting invariants (see [**5**]). The main result is the following:

THEOREM. *There is a natural group structure on A^n (respectively, B^n) which makes A^n (respectively, B^n) a finitely generated abelian group of which the torsion part consists of all tangential homotopy complex projective spaces (respectively, tangential homotopy quaternion projective spaces) and* rank $A^n = [(n + 1)/4] - 1$ *if n is odd or* $[(n + 1)/4]$ *if n is even (respectively,* $[n/2] - 1$). *Furthermore,* $s_{2k}: A^n \to L_{2k}(e)$ and $s_{4k}: B^n \to Z$ are *homomorphisms.*

REMARK. The computations of torsions of A^n or B^n are reduced to the computations of $[CP^{n-1}, F]$ or $[QP^{n-1}, F]$.

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Applications. Since ker s_{2k} consists of all free S^1 actions on homotopy $(2n-1)$-spheres with characteristic $(2k-1)$-spheres; hence

COROLLARY 1. *There exist infinitely many topologically inequivalent free S^1 actions on homotopy $(2n-1)$-spheres with characteristic homotopy $(2k-1)$-spheres for $n \geqq k \geqq 4$ and* (a) *$n \geqq 7$ if n is odd,* (b) *$n \geqq 6$ if both n and k are even and* (c) *$n \geqq 8$ if n is even and k is odd.*

REMARK. A similar result has been obtained by H.-T. Ku (see also [**1**]) using the techniques of W. C. Hsiang (see [**2**]). Our methods are different from theirs and the dimensions are much better than one can get from their papers.

Recall a theorem in [**6**]: An S^3 action is decomposable if and only if its restriction to S^1 is decomposable. Now rank $B^n <$ rank A^{2n}, and we have the following:

COROLLARY 2. *For $n \geqq 3$, there are infinitely many topologically inequivalent free S^1 actions on homotopy $(4n-1)$-spheres which are not extendable to free S^3 actions.*

Sketch of the proofs. Let $D^{n,q}$ be the group of pseudo-isotopy classes of diffeomorphisms of $K^{n,q}$ which are homotopic to the identity. Then the construction which is described in the Introduction yields a well-defined map $\psi: D^{n,q} \to A^{n,q}$. Using the techniques of G. R. Livesay and C. B. Thomas [**4**] (see also [**6**]) we can show

PROPOSITION 1. *ψ is onto.*

Consider the following exact sequence of Wall-Sullivan

$$0 \to \mathrm{hS}(K^{n,q} \times I, \partial(K^{n,q} \times I)) \xrightarrow{d} [\Sigma K^{n,q}, G/O] \xrightarrow{s} L_{2n-2}(e).$$

where $\mathrm{hS}(K^{n,q} \times I, \partial(K^{n,q} \times I))$ is the set of homotopy smoothings of $K^{n,q} \times I$ relative to the boundaries. It is well-known that there is a group structure on $\mathrm{hS}(K^{n,q} \times I, \partial(K^{n,q} \times I))$ such that d and s are homomorphisms. In our case $\mathrm{hS}(K^{n,q} \times I, \partial(K^{n,q} \times I))$ is abelian since $[\Sigma K^{n,q}, G/O]$ is abelian. Define a map b: $\mathrm{hS}(K^{n,q} \times I, \partial(K^{n,q} \times I)) \to D^{n,q}$ as follows. Let $h: M \to K^{n,q} \times I$ be a homotopy equivalence such that $h|\partial M: \partial M \to \partial(K^{n,q} \times I)$ is a diffeomorphism; $\partial M = \partial_0 M - \partial_1 M$. Let $h_0 = h|\partial_0 M$ and $h_1 = h|\partial_1 M$. Then set $b(M, h) = h_0 \cdot h_1^{-1}$.

PROPOSITION 2. *b is an epimorphism and* ker b *is finite.*

THEOREM 3. *$D^{n,q}$ is an abelian group and* rank $D^{n,q} =$ rank $H^{4*-1}(K^{n,q}, Q) - r^n$ *where $r^n = 1$ if n is odd and $r^n = 0$ if n is even.*

For $\alpha \in D^{n,q}$, let $s_{2k}(\alpha) = s_{2k}(P(\alpha))$.

THEOREM 4. *s_{2k} are homomorphisms and s_{4k} are nontrivial for $2q \leqq 4k < 2n$.*

This is proved by studying the following two commutative diagrams,

$$\begin{array}{ccccccc} hS(K^{n,q} \times I, \partial(K^{n,q} \times I)) & \to & [\Sigma K^{n,q}, G/O] & \to & [\Sigma K^{k,s}, G/O] & \searrow & \\ \downarrow & & \downarrow & & \downarrow & & L_{2k-2}(e) \\ hS(CP^{n-1}) & \longrightarrow & [CP^{n-1}, G/O] & \to & [CP^{k-1}, G/O] & \nearrow & \end{array}$$

and

$$\begin{array}{ccc} hS(K^{n,q} \times I, \partial(K^{n,q} \times I)) & \longrightarrow & \\ \downarrow b & \searrow & hS(CP^{n-1}) \to L_{2k-2}(e) \\ D^{n,q} & \longrightarrow & \end{array}$$

where the maps in diagrams are defined in the obvious way.

THEOREM 5. *For $\alpha \in D^{n,q}$, α is of finite order if and only if $P(\alpha)$ is tangential homotopy equivalent to CP^{n-1}.*

This follows from the fact that a homotopy complex projective space P is tangential equivalent to CP^{n-1} if and only if $s_{4k}(P) = 0$ for all k.

THEOREM 6. *For $n/2 \leqq q \leqq 2n/3$, there is a group structure on $A^{n,q}$ defined by $P(\alpha) * P(\beta) = P(\alpha \cdot \beta)$ and $\psi : D^{n,q} \to A^{n,q}$ is an epimorphism.*

Let $G^{n,q}$ be the subgroup of $D^{n,q}$ which is generated by those diffeomorphisms of $K^{n,q}$ which are extendable to either diffeomorphisms of $S^{2p-1} \times_{S^1} D^{2q}$ or diffeomorphisms of $D^{2p} \times_{S^1} S^{2q-1}$.

THEOREM 8. *For $n/2 \leqq q \leqq 2n/3$, $\ker \psi = G^{n,q}$.*

THEOREM 9. *For $n/2 \leqq q \leqq 2n/3$, $G^{n,q}$ is a finite group* (*see* [**6**]).

Finally,

THEOREM 10. *For $n/2 \leqq q \leqq 2n/3$, $A^{n,q}$ is a finitely generated abelian group and* rank $A^{n,q}$ = rank $D^{n,q}$.

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