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# SPACES HOMEOMORPHIC TO $(2^\alpha)_\alpha$

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Topological characterizations and Baire-category types of properties are obtained for the spaces $(2^\alpha)_\alpha$ of all infinite regular cardinals $\alpha$; in particular the stability of the class of spaces homeomorphic to $(2^\alpha)_\alpha$ when taking intersections of at most $\alpha$ open and dense subsets of $(2^\alpha)_\alpha$ is proved (Theorems 1, 2). Our interest in studying these spaces lies with applications of these results to questions about ultrafilters (and the Stone-Čech compactification of certain spaces) (Theorems 4, 5).

Detailed proofs of the results described in this note will appear elsewhere.

By a space we mean a completely regular Hausdorff topological space. A space $X$ is a $P_\alpha$-space if the intersection of every family of less than $\alpha$ open subsets of $X$ is an open subset of $X$; an element $p$ of $X$ is a $P_\alpha$-point if the intersection of every family of less than $\alpha$ neighborhoods of $p$ is a neighborhood of $p$. A family $\mathscr{B}$ of open subsets of $X$ is an $\alpha$-subbase of $X$ if the family $\mathscr{C}$ consisting of all intersections of families of elements of $\mathscr{B}$ of cardinality less than $\alpha$ is a base of $X$. For any space $X$, and any infinite regular cardinal $\alpha$, two $P_\alpha$-spaces are defined as follows: $P_\alpha(X)$ is the (possibly empty) subspace of $X$ consisting of all the $P_\alpha$-points of $X$; and $X_\alpha$ is the space with underlying set equal to the underlying set of $X$ and its topology defined by the requirement that the topology of $X$ is an $\alpha$-subbase for the topology of $X_\alpha$.

A space $X$ is $\alpha$-compact if every open cover of $X$ has a subcover of cardinality less than $\alpha$. A regular cardinal $\alpha$ is weakly compact if the space $(2^\alpha)_\alpha$ is $\alpha$-compact. A weakly compact cardinal is strongly inaccessible.

If $\mathscr{B}$ is a family of closed nonempty subsets of a space $X$, we say that $X$ is *$\mathscr{B}$-compact* if whenever $\mathscr{C} \subset \mathscr{B}$ and $\mathscr{C}$ has the finite intersection property (i.e., every finite subset of $\mathscr{C}$ has nonempty intersection), then $\bigcap \mathscr{C} \neq \emptyset$.

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An open partition $\mathscr{C}$ of a space $X$ is a family of open pairwise disjoint nonempty subsets of $X$ whose union is $X$.

We set $\alpha^{<\beta} \sum\{\alpha^\gamma = \gamma < \beta\}$.

**Theorem** 1. *Let $\alpha$ be an infinite regular cardinal. Let $X$ be a $P_\alpha$-space without isolated elements and with an $\alpha$-subbase $\mathscr{A}$ such that $|\mathscr{A}| \leqq 2^{<\alpha}$, $X$ is $\mathscr{A}$-compact, $\mathscr{A} = \bigcup_{\xi<\alpha} \mathscr{A}_\xi$, and $\mathscr{A}_\xi$ is an open partition of $X$ for $\xi < \alpha$. Then*

(a) *If $\alpha$ is not a weakly compact cardinal, then $X$ is homeomorphic to $(2^\alpha)_\alpha$.*

(b) *If $\alpha$ is a weakly compact cardinal, then $X$ is homeomorphic to $(2^\alpha)_\alpha$ if and only if $|\mathscr{A}_\xi| < \alpha$ for $\xi < \alpha$.*

**Theorem** 2. *Let $\alpha$ be an infinite regular cardinal.*

(a) *If $\alpha$ is not weakly compact, the intersection of any family of at most $\alpha$ open and dense subsets of $(2^\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.*

(b) *The intersection $G$ of any family of at most $\alpha$ open subsets of $(2^\alpha)_\alpha$ such that $G$ and $(2^\alpha)_\alpha \backslash G$ are both dense in $(2^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$.*

**Remarks.** Theorem 1 is proved by first using the properties of the $\alpha$-subbase to define a base that forms (in addition) a ramification system of order $\alpha$ under reverse set inclusion and whose set of elements of order $\xi$ is an open partition of $X$ for $\xi < \alpha$. In case $\alpha$ is not weakly compact this system can then be modified (by considering separately the partially overlapping cases where $\alpha$ is not strongly inaccessible and where $\alpha = \alpha^{<\alpha}$) into one in which every element has exactly $2^{<\alpha}$ immediate successors. In case $\alpha$ is weakly compact we proceed by an alternating chain argument. In both cases there results an isomorphism between these ramification systems (for two spaces satisfying conditions of Theorem 1) which is then easily translated into a homeomorphism of the spaces.

Theorem 2 is proved with similar techniques, using Theorem 1 and the fact (due to Sikorski [**15**]) that in $(2^\alpha)_\alpha$ the intersection of any family of at most $\alpha$ open and dense subsets of $(2^\alpha)_\alpha$ is dense. The assumption on the nonweak compactness of $\alpha$ in Theorem 2(a) is essential.

For $\alpha = \omega^+ = 2^\omega$ Theorem 1(a) is essentially proved by Parovičenko. Theorem 2(b) is proved in Kuratowski [**10**] for $\alpha = \omega$ and attributed there to Mazurkiewicz.

We note the following direct consequences of Theorem 1.

(a) If $\alpha \leqq \beta_\xi \leqq 2^{<\alpha}$ for $\xi < \alpha$ then the space $(\prod_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$. In particular, if $\alpha \leqq \beta \leqq 2^{<\alpha}$ then $(\beta^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$; thus $((2^{<\alpha})^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$.

(b) If $2 \leqq \beta_\xi < \alpha$ for $\xi < \alpha$ then the space $(\prod_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$. In particular, if $2 \leqq \beta < \alpha$, then $(\beta^\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.

(c) If $\alpha$ is not weakly compact and if $2 \leqq \beta_\xi \leqq 2^{<\alpha}$ for $\xi < \alpha$ then the space $(\prod_{\xi<\alpha} \beta_\xi)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$ (and also to $(\alpha^\alpha)_\alpha$).

(d) $\alpha$ is weakly compact if and only if $(2^\alpha)_\alpha$ is not homeomorphic to $(\alpha^\alpha)_\alpha$.

A useful consequence of Theorem 1 is the following

THEOREM 3. *Let $\alpha$ be an infinite regular cardinal and let $X$ be a compact (and totally disconnected if $\alpha = \omega$) space. If the weight of $X$ is $\alpha$ and the local weight (= character) of every element of $X$ is $\alpha$ then $X_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$. The converse holds if $\alpha = \alpha^{<\alpha}$.*

For an infinite cardinal $\alpha$ we denote by $U(\alpha)$ the set of uniform ultrafilters on $\alpha$ (considered as a subspace of the Stone-Čech compactification $\beta\alpha$), by $\mathrm{RK}(\alpha)$ the subspace of Rudin-Keisler minimal ultrafilters (cf. [**1**], [**2**], [**9**], [**14**]), and by $G(\alpha)$ the subspace of $\alpha^+$-good ultrafilters (cf. [**7**], [**9**]). Finally for a space $X$ we let $R(\beta X\backslash X)$ denote the set of remote points of $\beta X$ (cf. [**4**]. [**16**]).

The principal applications of the above results to spaces of ultrafilters are given in the next theorem.

THEOREM 4. *Let $\alpha$ be an infinite cardinal.*

(a) *If $2^\alpha$ is a regular cardinal and no uniform ultrafilter on $\alpha$ has a filter base of cardinality less than $2^\alpha$, then $(U(\alpha))_{2^\alpha}$ is homeomorphic to $(2^{(2^\alpha)})_{2^\alpha}$.*

(b) *If $\alpha^+ = 2^\alpha$, then each of the spaces $(U(\alpha))_{\alpha^+}$, $(\mathrm{RK}(\alpha))_{\alpha^+}$, $(G(\alpha))_{\alpha^+}$, $(\mathrm{RK}(\alpha) \cap G(\alpha))_{\alpha^+}$ is homeomorphic to $(2^{(\alpha^+)})_{\alpha^+}$.*

THEOREM 5. *Assume $\omega^+ = 2^\omega$. If $X$ is a noncompact locally compact realcompact space such that $|C(X)| \leqq 2^\omega$, then each of the spaces $(\beta X\backslash X)_{\omega^+}$ and $P_{\omega^+}(\beta X\backslash X)$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$. If $X$ is (in addition) a separable metric space without isolated elements, then each of the spaces $(R(\beta X\backslash X))_{\omega^+}$ and $R(\beta X\backslash X) \cap P_{\omega^+}(\beta X\backslash X)$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$.*

REMARKS. Theorem 3 has been proved for $\alpha = \omega^+ = 2^\omega$ in Theorem 3.2 of [**3**]. We note the following consequences: (a) If $\alpha = \alpha^{<\alpha}$ then $S(\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$, where $S(\alpha)$ denotes the Stone space of the $\alpha$-homogeneous-universal Boolean algebra of cardinality $\alpha$ (cf. [**11**]); (b) if $\alpha = \alpha^{<\alpha} > \omega$ then $P_\alpha(\Lambda(\alpha))$ and $P_\alpha(S(\alpha))$ are homeomorphic to $(\alpha^\alpha)_\alpha$, where $\Lambda(\alpha)$ denotes the set $2^\alpha$ with the lexicographic order topology (cf. [**13**]).

Theorem 4(a) is a direct application of Theorem 1. Theorem 4(b) is proved using Theorems 8.1 and 8.3 of Blass [**1**] and Theorem 2. The existence of $\alpha^+$-good ultrafilters on $\alpha$ was first proved by Keisler in [**7**] assuming $\alpha^+ = 2^\alpha$, and by Kunen [**9**] in general. The existence of $\alpha^+$-good ultrafilters on $\alpha$ which in addition are $P(\alpha)$-points (not defined here) was proved in [**12**], assuming $\alpha^+ = 2^\alpha$. A slight modification of this argument yields the existence of $\alpha^+$-good and uniformly selective ultrafilters on $\alpha$. A Baire category type of proof on the existence of $\alpha^+$-good and uniformly selective ultrafilters on $\alpha$, still assuming $\alpha^+ = 2^\alpha$, was first given by Blass in [**1**].

Most of the first half of Theorem 5 has been proved in [**3**]; specifically, Corollary 3.3 of [**3**] states that $(\beta X\backslash X)_{\omega^+}$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$; and Theorem 3.5 of [**3**] states that $P_{\omega^+}(\beta X\backslash X)$ is homeomorphic to $(2^{(\omega^+)})_{\omega^+}$ under certain assumptions on $X$ which are stronger than those assumed in the statement of the present Theorem 5.

Most of Theorem 5 can be proved assuming Martin's axiom instead of the continuum hypothesis (cf. [**2**] for the relation of Martin's axiom to ultrafilters on $\omega$).

The following results should be contrasted: If $\omega^+ = 2^\omega$ then $\beta\omega\backslash\omega$ is homeomorphic to the Stone space of the $2^\omega$-homogeneous-universal Boolean algebra $S(2^\omega)$ of cardinality $2^\omega$ (cf. Parovičenko [**13**] and Keisler [**8**]). However, if the continuum hypothesis fails then $\beta\omega\backslash\omega$ is definitely not homeomorphic to $S(2^\omega)$, according to a result of Hausdorff [**5**],[2] even though if Martin's axiom holds as well, then the results described above imply that $(\beta\omega\backslash\omega)_{2^\omega}$ *is* homeomorphic to $(S(2^\omega))_{2^\omega}$.

## BIBLIOGRAPHY

**1.** A. Blass, *Orderings of ultrafilters*, Ph.D. Thesis, Harvard University, Cambridge, Mass., 1970.

**2.** D. Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970/71), no. 1, 1–24. MR **43** #3104.

**3.** W. W. Comfort and S. Negrepontis, *Homeomorphs of three subspaces of $\beta N\backslash N$*, Math. Z. **107** (1968), 53–58. MR **38** #2739.

**4.** N. J. Fine and L. Gillman, *Remote points in $\beta R$*, Proc. Amer. Math. Soc. **13** (1962), 29–36. MR **26** #732.

**5.** F. Hausdorff, *Summen von $\aleph_1$ Mengen*, Fund. Math. **26** (1936), 241–255.

**6.** H. H. Hung, *The amalgamation property for G-metric spaces and homeomorphs of the space* $(2^\alpha)_\alpha$, Ph. D. Thesis, McGill University, Quebec, 1972.

**7.** H. J. Keisler, *Good ideals in fields of sets*, Ann. of Math. (2) **79** (1964), 338–359. MR **29** #3383.

**8.** ———, *Universal homogeneous Boolean algebras*, Michigan Math. J. **13** (1966), 129–132. MR **33** #3968.

**9.** K. Kunen, *Ultrafilters and independent sets* (manuscript).

**10.** K. Kuratowski, *Topologie*. Vol. 1, PWN, Warsaw, 1958; English transl., Academic Press, New York; PWN, Warsaw, 1966. MR **19**, 873; MR **36** #840.

**11.** S. Negrepontis, *The Stone space of the saturated Boolean algebras*, Trans. Amer. Math. Soc. **141** (1969), 515–527. MR **40** #1311.

**12.** ———, *The existence of certain uniform ultrafilters*, Ann. of Math. (2) **90** (1969), 23–32. MR **40** #46.

**13.** I. I. Parovičenko, *A universal bicompact of weight* $\aleph$, Dokl. Akad. Nauk SSSR **150** (1963), 36–39 = Soviet Math. Dokl. **4** (1963), 592–595. MR **27** #719.

**14.** M. E. Rudin, *Partial orders on the types in $\beta N$*, Trans. Amer. Math. Soc. **155** (1971), 353–362. MR **42** #8459.

**15.** R. Sikorski, *Remarks on some topological spaces of high power*, Fund. Math. **37** (1950), 125–136. MR **12**, 727.

**16.** R. G. Woods, *A Boolean algebra of regular closed subsets of $\beta X\backslash X$*, Trans. Amer. Math. Soc. **154** (1971), 23–36. MR 42 #5230.

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