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STABILITY IMPLIES NORMAL AND DISC BUNDLES[1]

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Introduction. In this note we announce results concerning normal bundles, disc bundles, and Stiefel-Whitney classes in the topological category. Many of these results also hold in the piecewise linear (PL) category, but the dimensions should be restricted accordingly.

I would like to thank Professors R. F. Brown and R. Edwards for many helpful and encouraging discussions.

Normal bundles and disc bundles. Let TOP_n be the semisimplicial (s.s.) group of topological origin-preserving homeomorphisms of R^n. Let $\mathrm{TOP}_{n,k}$ be the s.s. group of topological homeomorphisms of $R^n = R^{n-k} \times R^k$ which are pointwise fixed on R^k.

In [**9**] Kirby and Siebenmann announced a strong stability theorem for TOP/O, i.e., if $n \geqq 5$, the stability map

$$s_i\colon \pi_i(\mathrm{TOP}_n, O_n) \to \pi_i(\mathrm{TOP}_{n+1}, O_{n+1})$$

is an isomorphism for $i \leqq n + 1$ and an epimorphism for $i = n + 2$, where O_n is the s.s. n-dimensional orthogonal group. Using this result we deduce that

THEOREM 1. *$\pi_i(\mathrm{TOP}_n, \mathrm{TOP}_n(I)) = 0$ for $i \leqq n + 1, n \geqq 6$, where $\mathrm{TOP}_n(I)$ is the s.s. group of topological origin-preserving homeomorphisms of the unit disc in R^n.*

An immediate corollary is

COROLLARY 2. *Let X have the homotopy type of a k-dimensional CW complex. Any R^n-bundle over X contains a disc bundle if $n \geqq k - 2, n \geqq 6$. It is uniquely determined (up to isomorphism) if $n \geqq k - 1, n \geqq 6$.*

In particular, every n-manifold, $n \geqq 6$, has a tangent disc bundle.

Using the above stability result and results of Rourke and Sanderson ([**12**], [**13**]), we show that

THEOREM 3. *$\pi_i(\mathrm{TOP}_{n,k}, \mathrm{TOP}_{n-k}) = 0$ if $i \leqq n - k + j, n - k \geqq 5 + j$, $j = 0, 1, 2$.*

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Then, by the results of K. Millett [**10**], Browder [**2**], and Theorem 3, one can show the existence of disc bundles in Corollary 2 for R^n-bundles with nonzero cross section when $n = k - 3, n \geqq 11$.

Using Theorem 3 and the immersion theorem of Lees or Gauld [**7**] we immediately obtain

COROLLARY 4. *Let M^n be a locally flat submanifold of N^{n+q}. If $q \geqq 5 + i$, then M has a normal open or microbundle in N if $q \geqq n - i - 1, i = 0, 1, 2$. If $q \geqq n - i$ it is unique up to isotopy.*

REMARK. We obtain three more dimensions for disc bundles and open normal bundles than [**13**].

Stiefel-Whitney classes. Let $\xi = (E, E_0, p, B)$ be an R^n-bundle with zero-section, where $E_0 = E$-zero section, $p: E \to B$, and B is compact having the homotopy type of an m-dimensional CW-complex. Let $w_i(\xi)$ denote the ith *Stiefel-Whitney class* of ξ. The *top class of* ξ, denoted $W_n(\xi)$, is given by $W_n(\xi) = \Phi^{-1}(U_\xi \cup U_\xi)$, where $U_\xi \in H^n(E, E_0; p^*\Gamma)$ is the Thom class, Γ the local system of B such that

$$\Gamma(b) = H_n(p^{-1}(b), p_0^{-1}(b); Z),$$

and Φ the Thom isomorphism.

Let $\xi(V'_{n,k})$ be the bundle associated to ξ with fiber $\mathrm{TOP}_n/\mathrm{TOP}_{n-k}$. Let $c_{n-k+1}(\xi)$ denote the primary obstruction to finding a cross section of $\xi(V'_{n,k})$. Using Theorem 1, results of Akiba [**1**], and variations of classical techniques in the differentiable category we prove

THEOREM 5. *If $n \neq 4, 5, n \geqq m - 2$, then $c_n(\xi) = 0$ iff $W_n(\xi) = 0$. If $n - k \geqq 5, c_{n-k+1}(\xi)$ reduced* mod 2 *equals $w_{n-k+1}(\xi)$. Also $c_n(\xi) = \lambda W_n(\xi)$ where $\lambda = 0$ iff $W_n(\xi) = 0$.*

Let $\xi(V_{n,k})$ be the bundle associated to ξ with fiber $\mathrm{TOP}_n/\mathrm{TOP}_{n,k}$. Let $d_{n-k+1}(\xi)$ denote the primary obstruction to finding a cross section to $\xi(V_{n,k})$. Using a theorem of K. C. Millett [**10**] and Theorem 3, we show that

THEOREM 6. *If $n \neq 4, m + 5 \leqq 2(n - k)$, then $d_n(\xi) = 0$ iff $W_n(\xi) = 0$. If $n - k \geqq 5$, then $d_{n-k+1}(\xi)$ reduced* mod 2 *equals $w_{n-k+1}(\xi)$. Also $d_n(\xi) = \lambda W_n(\xi)$ where $\lambda = 0$ iff $W_n(\xi) = 0$.*

These theorems provide a geometrical interpretation of the Stiefel-Whitney classes in the topological category similar to that in the differentiable category (see [**11**], [**14**]).

REMARK. We can also prove results analogous to Theorems 4 and 5 in the PL category with no restrictions on $n - k$, by using results of Hirsch [**8**]. However, in Theorem 5 we must have $n \geqq m$.

Let M^n be a compact n-manifold with or without boundary. An arc-field on M is a map $p:M \to M^I$ such that, for all $b \in M$, $p(b)(0) = b$ and $p(b)$ is a homeomorphism. Using results of R. F. Brown and E. Fadell [**3**], Hirsch [**8**], and Theorem 6 we make the following observation.

COROLLARY 7. *If M^n is a PL-manifold, or if $n \neq 4$, M has an arc-field iff the Euler characteristic of M is zero.*

This was proven for all triangulated manifolds by Fadell [**5**].

REMARK. In view of Theorems 5 and 6 the appropriate topological version of a tangent k-field should be a map $p:M \to M^{R^k}$ where, for all $b \in M$, $p(b)(0) = b$ and $p(b)$ is a locally flat homeomorphism. This answers a question posed by E. Fadell [**6**].

Details of proofs will appear elsewhere.

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