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PONTRJAGIN CLASSES OF PL SHEAVES

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ABSTRACT. Over the category of PL manifolds there is a fibered category whose objects are certain equivalence classes $[\mathscr{F}]$ of "PL sheaves" $\mathscr{F}$, to which one assigns real characteristic classes as in [2] and [3]. In particular each PL manifold M possesses a distinguished (co)tangent object $[\mathscr{E}(M)]$ and a real Pontrjagin class $p([\mathscr{E}(M)])$. In this note we show that $p([\mathscr{E}(M)])$ is the image under $H^{4*}(M;Q) \to H^{4*}(M;R)$ of the Thom-Pontrjagin class of M.

The construction of [2] and [3] assigns total Chern classes $c([\mathscr{F}]) \in H^{2*}(M;R)$ to cosets $[\mathscr{F}]$ of complex PL sheaves $\mathscr{F}$ over a PL manifold M, and this assignment satisfies certain axioms. As in the classical case one defines the total Pontrjagin class $p([\mathscr{F}]) \in H^{4*}(M;R)$ of a coset $[\mathscr{F}]$ of real PL sheaves via complexification of $[\mathscr{F}]$, and here are the corresponding axioms:

(P_1) if $[\mathscr{F}]$ is a coset of real PL sheaves of "rank" m on a PL manifold M then the total Pontrjagin class $p([\mathscr{F}])$ is an element $1 + p_1([\mathscr{F}]) + \cdots + p_{[m/2]}([\mathscr{F}])$ of $H^*(M;R)$ with $p_i([\mathscr{F}]) \in H^{4i}(M;R)$;

(P_2) $p(\Xi^![\mathscr{F}]) = \Xi^* p([\mathscr{F}]) \in H^{4*}(N;R)$ for any PL map $\Xi: N \to M$;

(P_3) $p([\mathscr{F}] \oplus [\mathscr{G}]) = p([\mathscr{F}]) \cup p([\mathscr{G}])$ for any cosets $[\mathscr{F}]$ and $[\mathscr{G}]$ over M;

(P_4) if $[\mathscr{F}]$ contains a bona fide real vector bundle ξ over M (as in [2] or [3]) then $p([\mathscr{F}])$ is the classical total Pontrjagin class $p(\xi) \in H^{4*}(M;R)$.

LEMMA 1. *If a PL manifold M happens to admit a smooth structure with tangent bundle* τ_M *then* $p([\mathscr{E}(M)]) = p(\tau_M)$.

PROOF. One easily verifies as in [2] that $[\mathscr{E}(M)]$ contains τ_M; hence it suffices to apply (P_4).

As in the smooth case one uses axioms (P_1), (P_2), (P_3) and the multiplicative sequence corresponding to $z^{1/2}/(\tanh z^{1/2})$ to construct the Hirzebruch polynomial $l([\mathscr{F}]) \in H^{4*}(M; R)$ of the Pontrjagin class $p([\mathscr{F}])$,

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and one defines the L-genus $L(M)$ of any compact oriented PL manifold M of dimension $4n$ by setting $L(M) = \langle l_n([\mathscr{E}(M)]), [M] \rangle$ for the fundamental class $[M] \in H_{4n}(M;R)$. The index $I(M)$ is defined as usual, and the classical Hirzebruch index formula combines with Lemma 1 to guarantee that $L(M) = I(M)$ whenever the PL manifold M happens to have a smooth structure.

LEMMA 2. *$L(M) = I(M)$ for any compact oriented PL manifold M of dimension $4n$.*

PROOF. One easily verifies as in the smooth case that

$$L(M+N) = L(M) + L(N),\ L(-M) = -L(M),\ L(M \times N) = L(M)\cdot L(N)$$

and $L(\text{boundary}) = 0$, so that L may be regarded as a homomorphism $\Omega_*^{\mathrm{PL}} \otimes R \to R$. Hence it will suffice to verify that the homomorphism L agrees on generators with the corresponding homomorphism $I:\Omega_*^{\mathrm{PL}} \otimes R \to R$. But the homomorphism $\Omega_* \to \Omega_*^{\mathrm{PL}}$ is injective (by (P_4), for example), which yields an exact sequence

$$0 \to \Omega_q \to \Omega_q^{\mathrm{PL}} \to \Omega_q^{\mathrm{PL}}/\Omega_q \to 0$$

in each dimension q, and Williamson showed in [**5**] that each $\Omega_q^{\mathrm{PL}}/\Omega_q$ is finite. Hence $\Omega_* \otimes R \to \Omega_*^{\mathrm{PL}} \otimes R$ is an isomorphism, so that each class in $\Omega_*^{\mathrm{PL}} \otimes R$ contains at least one smooth manifold; but we already know from Lemma 1 that $L(M) = I(M)$ for smooth manifolds M.

Now for any PL manifold M let $P(M) \in H^{4*}(M;R)$ be the image under $H^*(M;Q) \to H^*(M;R)$ of the rational Pontrjagin class constructed by Thom in [**4**]. (See [**1**] for an alternate version of Thom's construction.) Thom's construction established the existence of unique rational classes satisfying certain axioms, which we translate as follows into real cohomology:

(T_1) for each oriented PL manifold M of dimension m the class $P(M)$ is of the form $1 + P_1(M) + \cdots + P_{[m/2]}(M)$ with $P_i(M) \in H^{4i}(M;R)$;

(T_2) to each embedding $i:N \to M$ of one oriented PL manifold into another one can assign a "normal" class $Q(N) \in H^{4*}(N;R)$ satisfying $P(N) \cup Q(N) = i^*P(M)$;

(T_3) if $l'(M) \in H^{4*}(M;R)$ is the Hirzebruch polynomial constructed from $z^{1/2}/(\tanh z^{1/2})$ and $P(M)$ then the L'-genus defined for any $4n$-dimensional compact oriented PL manifold M by $L'(M) = \langle l'_n(M), [M] \rangle$ satisfies $L'(M) = I(M)$.

Here is the main result of this note, which permits one to conclude that the Pontrjagin classes $p([\mathscr{F}])$ form an extension of the Thom-Pontrjagin construction to a reasonable fibered category over the category of PL manifolds:

PROPOSITION. $p([\mathscr{E}(M)]) = P(M) \in H^{4*}(M;R)$ *for any PL manifold* M.

PROOF. It suffices to verify that the classes $p([\mathscr{E}(M)])$ satisfy Thom's axioms. But (T_1) is an immediate consequence of (P_1), (T_2) follows easily from (P_2) and (P_3), and (T_3) follows from Lemma 2.

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