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A CHARACTERIZATION OF GROWTH IN LOCALLY COMPACT GROUPS

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G will denote throughout a separable, connected, locally compact group. Fix a left Haar measure on G and for a measurable subset A of G, let $|A|_G$ denote the measure of A. The purpose of this note is to announce results concerning the asymptotic behavior of $|U^n|_G$ where U is a compact neighborhood of the identity e in G, and to indicate some of the applications these results have for various areas. The following definitions are required:

DEFINITION 1. G has *polynomial growth* if there is a polynomial p such that for each compact neighborhood U of e, there is a constant $C(U)$ so that

$$|U^n|_G \leqq C(U)p(n) \qquad (n = 1, 2, \ldots)$$

($U^n = \{u_1u_2, \ldots, u_n | u_i \in U, 1 \leqq i \leqq n\}$). G has *exponential growth* if for each compact neighborhood U of e there is a $t > 1$ such that

$$|U^n|_G \geqq t^n \qquad (n = 1, 2, \ldots).$$

Note that since G is connected, its "growth" will be determined by the behavior of $|U^n|_G$ for any one compact neighborhood U of e.

For $a, b \in G$, let $[a, b]$ denote the subsemigroup of G generated by a and b, i.e.,

$$[a, b] = \{x_1x_2, \ldots, x_n | x_i \in \{a, b\}, 1 \leqq i \leqq n, n = 1, 2, \ldots\}.$$

$[a, b]$ is said to be free if $a[a, b] \cap b[a, b] = \emptyset$. A subset S of G is uniformly discrete if there is a neighborhood U of e in G such that $sU \cap tU = \emptyset$ for $s, t \in S, s \neq t$.

DEFINITION 2. G is *type NF* if there does not exist $a, b \in G$ such that $[a, b]$ is free and uniformly discrete.

Let H be a connected Lie group with Lie algebra $\mathfrak{h}$, and let $g \to \mathrm{Ad}g$ be the canonical adjoint representation of H on $\mathfrak{h}$. H is said to be type R if the eigenvalues of $\mathrm{Ad}g$ are of absolute value one for each $g \in H$.

Since G is connected, there exists an arbitrarily small compact normal subgroup K of G such that G/K is a Lie group.

DEFINITION 3. G is *type R* if there exists a compact normal subgroup K

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such that G/K is a type R Lie group.

THEOREM 4. *The following conditions are equivalent*:
(i) *G has polynomial growth,*
(ii) *G is type NF,*
(iii) *G is type R.*

OUTLINE OF PROOF. (i) $\Rightarrow$ (ii) is straightforward. To establish that (ii) $\Rightarrow$ (iii), we define groups G_θ for each $\theta = \theta_1 + i\theta_2, \theta_1, \theta_2 \in \boldsymbol{R}, \theta_1 \neq 0$ and show that each G_θ is not type NF and that, if G is not type R, G contains some G_θ as a topological subgroup. It then follows that G is not type NF.

To show that (iii) $\Rightarrow$ (i), we first reduce to the case where G is simply connected and solvable. One can then write $G = g_1(t_1)g_2(t_2) \cdots g_n(t_n)$ where each $g_i(t_i)$ is a one parameter subgroup of G. The argument proceeds by induction on n, using the fact that since G is type R, $\|\mathrm{Ad}g_i(t)\| \leqq p(t)$ for some polynomial p.

Comparison with discrete groups. Milnor [**8**] and Wolf [**11**] have investigated the growth of discrete solvable groups in connection with the study of fundamental groups of Riemannian manifolds with negative curvature. Combining their results with a recent result of Tits [**10**], one has the following: If H is a linear group over a field k with a finite set of generators $A = A^{-1}$, then (i) either $|A^n|_H \leqq p(n)$ for some polynomial p and all $n \geqq 1$ or there is a $t > 1$ such that $|A^n|_H \geqq t^n$ for all $n \geqq 1$, and (ii) if $|A^n|_H$ has polynomial growth, then H is a finite extension of a solvable group S and S is a finite extension of a nilpotent group. We obtain analogous results for connected groups as a corollary to Theorem 4.

COROLLARY 5. (i) *Either G has polynomial growth or G has exponential growth.*

(ii) *If G is a connected Lie group with polynomial growth, then G is the compact extension of a solvable Lie group S and* Ad S *is an analytic subgroup of a compact extension of a nilpotent group.*

REMARK. The first part of this corollary shows that in a connected group, a compact set cannot grow at a rate intermediate to polynomial and exponential, for example, such as $t^n/\log n$. This answers a question raised in Emerson and Greenleaf [**4**]. With regard to the second part, we remark that Hulanicki [**5**] has shown that a separable, locally compact group that is the compact extension of a nilpotent group cannot have exponential growth.

Strong amenability. In [**4**] Emerson and Greenleaf define a locally compact group H to be strongly amenable if for every compact neighborhood $U = U^{-1}$ of e in H

$$\lim_n |U^{n+1}|_H/|U^n|_H = 1.$$

Greenleaf has asked if every connected, amenable, unimodular group is necessarily strongly amenable. The following corollary to Theorem 4 provides a large class of counterexamples.

COROLLARY 6. *If G is strongly amenable, then G is type R. If G is type R, then*

$$\liminf_n |U^{n+1}|_G/|U^n|_G = 1$$

for each compact neighborhood U of e.

In particular, let G be the semidirect product of $\boldsymbol{R}$ with $\boldsymbol{R}^2$ given by the homomorphism $\varphi:\boldsymbol{R} \to \mathrm{Aut}(\boldsymbol{R}^2)$ where $\varphi(t)(x, y) = (e^t x, e^{-t} y)$ for $t \in \boldsymbol{R}$, $(x, y) \in \boldsymbol{R}^2$. Then G is connected, amenable, unimodular but not type R, and hence, not strongly amenable.

An ergodic theorem. Let X be a compact, separable metric space and assume G is unimodular and has a jointly continuous action $G \times X \to X$ on X. A sequence of Borel subsets $\{A_n\}$ of G is called *balanced with respect to the action of G on X* if $0 < |A_n|_G < \infty$ for each n and if whenever μ is a probability measure on X invariant and ergodic under G and $f \in C(X)$, the continuous complex valued functions on X, then

$$\lim_n |A_n|_G^{-1} \int_{A_n} f(g \circ x_0)\, dg$$

exist and equals $\int f\, d\mu$ for μ-almost all $x_0 \in X$.

An increasing sequence of subsets $\{A_n\}$ of G *grows evenly* in G if $0 < |A_n|_G < \infty$ for each n,

$$\lim_k |A_k|_G^{-1} |(A_k A_n) \Delta A_k|_G = 0 = \lim_k |A_k|_G^{-1} |(A_n A_k) \Delta A_k|_G$$

for each n, and there is a constant $c > 0$ such that $|A_n^{-1} A_n|_G \leqq c|A_n|_G$ for each n.

Calderón [3] and Bewley [2] have proved the following generalization of Birkhoff's individual ergodic theorem: If G contains a sequence $\{A_n\}$ that grows evenly in G, then $\{A_n\}$ is balanced with respect to the action of G on X.

Auslander and Brezin [1] have shown that any connected, simply connected, nilpotent Lie group N contains a sequence of compact connected subsets that grow evenly in N. This is a special case of

COROLLARY 7. *If G satisfies the equivalent conditions of Theorem* 4 *and* *if $U = U^{-1}$ is a compact neighborhood of the identity, then a subsequence of $\{U^n | n = 1, 2, \ldots\}$ grows unevenly in G.*

On symmetry of $\mathscr{L}^1(G)$. A Banach *-algebra $\mathscr{U}$ is symmetric if $-xx^*$ is quasi-regular for each $x \in \mathscr{U}$, or equivalently by Raikov's Theorem [**9**], if

$$(\nu(x) =) \lim_n \|x^n\|^{1/n} = \sup \|T_x\| \qquad (= \lambda(x))$$

for each $x = x^* \in \mathscr{U}$, where the sup is taken over all *-representations $x \to T_x$ of $\mathscr{U}$. Hulanicki [**5**] has shown that if H is a separable, locally compact group such that $\lim_n |A^n|_H^{1/n} \leqq 1$ for any compact subset A of G, then $\nu(x) = \lambda(x)$ for all $x = x^* \in \mathscr{L}^1(H)$ with compact support. Thus, any group with polynomial growth "almost" has a symmetric group algebra. (Observe that symmetry fails in this case only when the spectral radius is not continuous, and it is not known if this can ever occur in a group algebra.)

On the other hand, if H is a discrete group, $l^1(H)$ is not symmetric if H contains a free semigroup $[a, b]$ (cf. Jenkins [**6**]). There is evidence that suggests a similar statement obtains if G is not type NF. Theorem 4, therefore, lends support to a conjecture this author originally stated in [**7**], to wit, $\mathscr{L}^1(G)$ is symmetric if, and only if, G is type NF.

Proofs of these and related results will appear elsewhere. This author wishes to express his thanks to R. Howe for many helpful suggestions related to this work.

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