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UNITARY WHITEHEAD GROUP OF CYCLIC GROUPS

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In topology, one encounters certain groups known as surgery obstruction groups, introduced by C. T. C. Wall [7]. They can be described in a purely algebraic setting, and so hopefully be computed by algebraic means, notably by techniques developed by H. Bass [1], which has come to be known as algebraic K-theory. This note aims at applying such techniques to the computation of the so-called unitary Whitehead groups, certain quotients of which are the Wall's surgery groups mentioned above. We only state the results and sketch the proof of the main theorem. Details will appear elsewhere. This work constitutes part of Chapter I of the author's dissertation [6] submitted to Columbia University. I am deeply indebted to my adviser, Professor Hyman Bass, for his extraordinary patience, generous help and inspiring guidance.

A unitary ring is a triple (A, λ, Λ), where A is a ring with involution denoted by $a \mapsto \bar{a}$, λ is an element in the center of A satisfying $\lambda\bar{\lambda} = 1$, and Λ is an additive subgroup of A satisfying the conditions

$$S_{-\lambda}(A) = \{a - \lambda\bar{a} | a \in A\} \subset \Lambda \subset \{a \in A | a + \lambda\bar{a} = 0\} = S^{-\lambda}(A)$$

and

$$\bar{a}ra \in \Lambda$$

whenever $a \in A, r \in \Lambda$. A morphism $f : (A, \lambda, \Lambda) \to (A', \lambda', \Lambda')$ between two unitary rings is a ring homomorphism $f : A \to A'$ satisfying the conditions $f(\lambda) = \lambda', f(\bar{a}) = \overline{f(a)}$ for all $a \in A$ and $f(\Lambda) \subset \Lambda'$. By an epimorphism (of unitary rings) we mean a morphism with $f(A) = A'$ and $f(\Lambda) = \Lambda'$. When A has trivial involution (that is, $a = \bar{a}$ for all $a \in A$, so that A has to be commutative), we single out the case $(A, 1, 0)$ and call it the orthogonal case. The symbol $U^{\lambda}_{2n}(A, \Lambda)$ will denote the group of all invertible $2n \times 2n$-matrices $\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ such that

$$\sigma^{-1} = \begin{pmatrix}\delta^* & \lambda\beta^* \\ \bar{\lambda}\gamma^* & \alpha^*\end{pmatrix}$$

where * means conjugate transpose, and $\beta\alpha^*, \delta\gamma^*$ have diagonal entries in Λ. The symbol $U^{\lambda}(A, \Lambda)$ will denote the group inj lim $U^{\lambda}_{2n}(A, \Lambda)$ with respect to the filtering given by

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$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The unitary Whitehead group, denoted by $KU_1^\lambda(A, \Lambda)$, is the commutator quotient group $U^\lambda(A, \Lambda)/[U^\lambda(A, \Lambda), U^\lambda(A, \Lambda)]$. It can be shown [**6**] that this group can also be described as the Whitehead group of a certain (λ, Λ)-unitary category. There is a homomorphism $H: K_1(A) \to KU_1^\lambda(A, \Lambda)$ induced by the association

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}.$$

The cokernel of H will be denoted by $W_1^\lambda(A, \Lambda)$. In the orthogonal case, we write KO_1, WO_1 instead of KU_1^1, W_1^1 respectively.

We are mainly interested in the case $A = Z\pi$, the integral group ring of a group π, with involution given by $g \mapsto \bar{g} = g^{-1}$ for all $g \in \pi$, and $\lambda = 1$ or -1, and $\Lambda = S_{-\lambda}(Z\pi)$. We shall denote $L_1(\pi) = W_1^1(Z\pi, S_{-1}(Z\pi))/\langle w_1 \rangle$ where w_1 is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and denote

$$L_3(\pi) = W_1^{-1}(Z\pi, S_1(Z\pi))/\langle w_{-1} \rangle$$

where w_{-1} is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The case $L_3(\pi)$ has been extensively investigated by Bass [3], starting from a computation by Lee [**5**] for the case π cyclic of odd prime order. The main result is $L_3(\pi) = 0$ for π finite abelian of odd order. In this note, we attempt to investigate the case $L_1(\pi)$.

THEOREM 1. *If π is cyclic of odd prime power order p^r, then*

$$W_1^1(Z\pi, S_{-1}(Z\pi))$$

is a group of order two, generated by w_1.

COROLLARY 2. *$L_1(\pi) = 0$ for π cyclic of odd prime power order.*

COROLLARY 3. *If π is cyclic of odd prime power order p^r, then*

$$KU_1^1(Z\pi, S_{-1}(Z\pi))$$

has exponent $2p^r$.

Two main tools in the computation are the Mayer-Vietoris sequence associated to a cartesian square and the unitary analogue of Milnor's K_2 group. For the latter, consider the group $EU^\lambda(A, \Lambda)$, which is the subgroup of $U^{\lambda}(A, \Lambda)$ generated by matrices of the form $X_{+}(\beta) = \left(\begin{smallmatrix} I & \beta \\ 0 & I \end{smallmatrix}\right)$ and $X_{-}(\gamma) = \left(\begin{smallmatrix} I & 0 \\ \gamma & I \end{smallmatrix}\right)$. It can be shown [**2**] that the group $EU^{\lambda}(A, \Lambda)$ is perfect, so that it possesses a universal central covering. The kernel of this universal central covering will be denoted by $KU_2^{\lambda}(A, \Lambda)$. In the orthogonal case, KU_2^1 is written KO_2 instead.

THEOREM 4. *Suppose*

$$\begin{array}{ccc} (A, \Lambda) & \xrightarrow{i_1} & (A_1, \Lambda_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_2} \\ (A_2, \Lambda_2) & \xrightarrow{j_1} & (A', \Lambda') \end{array}$$

is a cartesian square of unitary rings (λ is same throughout), with i_1, i_2, j_1, j_2 being epimorphisms of unitary rings. Then there is an exact sequence

$$KU_2^{\lambda}(A_1, \Lambda_1) \oplus KU_2^{\lambda}(A_2, \Lambda_2) \xrightarrow{j_2 - j_1} KU_2^{\lambda}(A', \Lambda') \xrightarrow{\partial} KU_1^{\lambda}(A, \Lambda)$$
$$\xrightarrow{(i_1, i_2)} KU_1^{\lambda}(A_1, \Lambda_1) \oplus KU_1^{\lambda}(A_2, \Lambda_2) \xrightarrow{j_2 - j_1} KU_1^{\lambda}(A', \Lambda').$$

This is called the Mayer-Vietoris sequence associated to the cartesian square.

In the computation, we use the following cartesian square ($\lambda = 1$)

$$\begin{array}{ccc} (Z\pi, \Lambda) & \xrightarrow{i_1} & (Z\pi^*, \Lambda^*) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_2} \\ (Z, 0) & \xrightarrow{j_1} & (Z/mZ, 0) \end{array}$$

where $Z\pi^* = Z\pi/(\Sigma)$ with $\Sigma = \sum_{g \in \pi} g$, $\Lambda^* = S_{-1}(Z\pi^*)$, $\Lambda = S_{-1}(Z\pi)$ and $m =$ order of π. The homomorphisms i_1, j_1 are the natural epimorphisms, the homomorphism i_2 is the augmentation map and the homomorphism j_2 sends $x + (\Sigma)$ to $i_2(x) + mZ$. The Mayer-Vietoris sequence associated to the cartesian square is as follows.

$$(*) \quad \begin{array}{l} KU_2^1(Z\pi^*, \Lambda^*) \oplus KO_2(Z) \xrightarrow{j_2 - j_1} KO_2(Z/mZ) \xrightarrow{\partial} KU_1^1(Z\pi, \Lambda) \\ \xrightarrow{(i_1, i_2)} KU_1^1(Z\pi^*, \Lambda^*) \oplus KO_1(Z) \xrightarrow{j_2 - j_1} KO_1(Z/mZ). \end{array}$$

The crux of the work lies in the investigation of the group $KU_1^1(Z\pi^*, \Lambda^*)$. By a certain device, we can pass partially into the $\lambda = -1$ case and obtain the following information.

PROPOSITION 5. *If π is cyclic of odd order m, then the homomorphism $j_2 : W_1^1(Z\pi^*, \Lambda^*) \to WO_1(Z/mZ)$ is injective.*

Concerning the KU_2^λ groups, we have the following.

PROPOSITION 6. *If π is cyclic of odd prime power order $m = p^r$, then the homomorphism $j_1 : KO_2(Z) \to KO_2(Z/mZ)$ is surjective.*

PROOF OF THEOREM 1. By Proposition 6, we may replace the group $KO_2(Z/mZ)$ in the exact sequence (*) by the trivial group. Thus we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & K_1(Z\pi) & \longrightarrow & K_1(Z\pi^*) \oplus K_1(Z) & \longrightarrow & K_1(Z/mZ) \longrightarrow 0 \\ & & \downarrow H & & \downarrow H & & \downarrow H \\ 0 & \longrightarrow & KU_1^1(Z\pi) & \longrightarrow & KU_1^1(Z\pi^*) \oplus KO_1(Z) & \longrightarrow & KO_1(Z/mZ) \end{array}$$

By the Snake Lemma, together with the observation that the homomorphism

$$\ker\{K_1(Z\pi^*) \oplus K_1(Z) \to KU_1^1(Z\pi^*) \oplus KO_1(Z)\}$$
$$\to \ker\{K_1(Z/mZ) \to KO_1(Z/mZ)\}$$

is surjective, we obtain a new exact sequence

$$0 \to W_1^1(Z\pi) \longrightarrow W_1^1(Z\pi^*) \oplus WO_1(Z) \longrightarrow WO_1(Z/mZ).$$

By Proposition 5, the homomorphism $j_2 - j_1$ restricted to $W_1^1(Z\pi^*)$ is injective. Thanks to Bass' work [4] on orthogonal groups, we see that $WO_1(Z)$ is generated by w_1. Hence the homomorphism $j_2 - j_1$ restricted to $W_1^1(Z\pi^*)$ has the same image as $j_2 - j_1$. Hence the kernel of $j_2 - j_1$ is isomorphic to $WO_1(Z)$. The result now follows.

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