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GROUPS OF INVERTIBLE ELEMENTS OF BANACH ALGEBRAS[1]

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ABSTRACT. Let A be a complex Banach algebra, G its group of invertible elements, and G_e the component of the identity of G. Then G_e is a closed, normal subgroup of G. This paper contains examples of B^* algebras A for which G/G_e is finite, but not trivial, and of a B^* algebra for which G/G_e is noncommutative.

Let A denote a complex Banach algebra, G its group of invertible elements, and G_e the component of the identity of G. If A is finite-dimensional, or if $A = B(H)$, the algebra of all bounded linear operators on a Hilbert space H, or if A is commutative, then G/G_e is torsion free. For the first two cases we actually have G connected, so $G = G_e$. A proof of the last result, which is due to Lorch, can be found in [**3**, p. 15]. We shall give examples of closed, noncommutative subalgebras of $B(H)$ for which G/G_e is finite, but not trivial, and of a B^* algebra for which G/G_e is not abelian. Our examples will be special cases of the following class of Banach algebras.

Let m be a finite, positive Borel measure whose support is a compact Hausdorff space X. Let $A(X, n)$ denote the set of all continuous functions from X into M_n, the algebra of all complex $n \times n$ matrices. Then $A(X, n)$ is a Banach algebra under the pointwise addition and multiplication of functions and the following norm:

$$\|F\| = \sup_{x \in X} |F(x)|, \qquad F \in A(X, n),$$

where

$$|F(x)| = \sup\left\{|F(x)y| : y \in \mathbf{C}^n, \sum_{i=1}^{n} |y_i|^2 \leqq 1\right\}.$$

We can also define an involution on $A(X, n)$ by

$$F^*(x) = (F(x))^* \quad \text{for } F \in A(X, n) \text{ and } x \in X,$$

where $(F(x))^*$ denotes the conjugate transpose of $F(x)$. Then $A(X, n)$ is a B^*-algebra under this norm and involution. For, each $F \in A(X, n)$ induces an operator $\tilde{F}$ on $H = L^2(m) \oplus \cdots \oplus L^2(m)$ by

$$(\tilde{F}f)(x) = F(x)f(x) \quad \text{for } f \in H \text{ and } x \in X.$$

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One checks that $(F^*)^\sim$ is the adjoint of $\tilde{F}$ and that $F \to \tilde{F}$ is a norm-preserving isomorphism of $A(X, n)$ onto a B^*-subalgebra of $B(H)$.

The group of invertible elements of $A(X, n)$, which we denote by $G(X, n)$, is the set of all continuous functions from X into $GL(n)$, the group of all invertible complex $n \times n$ matrices. Let $G_e(X, n)$ denote the component of the identity of $G(X, n)$. Then the elements of $G(X, n)/G_e(X, n)$ are just the homotopy classes of maps from X into $GL(n)$ because two maps $f, g \in G(X, n)$ are homotopic if and only if $fg^{-1} \in G_e(X, n)$.

Notation. Let S and T be topological spaces. We let $[S, T]$ denote the space of homotopy classes of maps from S into T with the topology derived from the compact-open topology on the set Map(S, T) of continuous functions from S into T. If T is a topological group, then so is Map(S, T) under the pointwise multiplication of functions, and it is easily shown that $[S, T]$ is also a topological group [**4**, p. 34].

PROPOSITION. *Let $A(X, n)$, $G(X, n)$, $G_e(X, n)$ be defined as above. Then $G(X, n)/G_e(X, n)$ is homeomorphic and isomorphic to $[X, \mathcal{U}(n)]$ where $\mathcal{U}(n)$ denotes the group of all complex unitary $n \times n$ matrices.*

PROOF. We have $G(X, n)/G_e(X, n) = [X, GL(n)]$ because $G(X, n)$ is Map$(X, GL(n))$ and two maps in $G(X, n)$ are homotopic if and only if they lie in the same component of $G(X, n)$. Now since $GL(n)$ is homeomorphic to $\mathcal{U}(n) \times \boldsymbol{R}^{n^2}$ [**1**, p. 16] and $\boldsymbol{R}^{n^2}$ is contractible, every $f \in$ Map$(X, GL(n)$ is homotopic to a map $\tilde{f} \in$ Map$(X, \mathcal{U}(n))$. Thus $[X, GL(n)]$ is homeomorphic and isomorphic to $[X, \mathcal{U}(n)]$ under the map $f \to \tilde{f}$.

We now construct our examples by choosing appropriate spaces X.

EXAMPLE 1. Let S^k denote the real k-sphere. Then the following statements hold for the algebra $A(S^k, n)$:

(a) $[S^k, \mathcal{U}(n)] \cong \pi_k(\mathcal{U}(n))$,

(b) $G(S^k, 2)/G_e(S^k, 2) \cong \pi_k(\mathcal{U}(2)) \cong \pi_k(S^3)$.

In particular, $\pi_k(S^3)$ is finite and nontrivial if $4 \leqq k \leqq 22$.

PROOF. Part (a) is true because both groups have the same elements, S^k is a suspension and $\mathcal{U}(n)$ is an H-space (see [**4**, p. 44]). The first equality in part (b) follows from the Proposition. The second equality in (b) holds because $\mathcal{U}(2)$ is homeomorphic to $S^3 \times S^1$ [**5**, p. 129] and so $\pi_k(\mathcal{U}(2)) = \pi_k(S^3) \oplus \pi_k(S^1)$. For $k > 1$, $\pi_k(S^1) = 0$ [**5**, p. 111] and if $4 \leqq k \leqq 22$, then $\pi_k(S^3)$ is finite and nontrivial [**6**, pp. 186–188] and so for these k at least $G(S^k, 2)/G_e(S^k, 2)$ is finite and nontrivial.

Note. The result may hold for other k as well, but these are the only values of $\pi_k(S^3)$ given in the reference quoted above.

The algebras $A(S^k, n)$ always have abelian factor groups $G(S^k, n)/G_e(S^k, n) \cong \pi_k(\mathcal{U}(n))$ because $\mathcal{U}(n)$ is an H-space [**4**, p. 44]. The following example of a complex Banach algebra with a nonabelian factor group G/G_e is due to E. Fadell.

EXAMPLE 2. Let $X = \mathscr{U}(2) \times \mathscr{U}(2)$. Then in the algebra $A(X, 2)$, $G(X, 2)/G_e(X, 2) \cong [X, \mathscr{U}(2)]$ is nonabelian.

PROOF. Define the following four maps on $\mathscr{U}(2) \times \mathscr{U}(2)$:

$$f(u, v) = u, \qquad \varphi(u, v) = (v, u), \qquad g(u, v) = v, \qquad \mu(u, v) = u \cdot v$$

where $(u, v) \in \mathscr{U}(2) \times \mathscr{U}(2)$ and the dot indicates multiplication in $\mathscr{U}(2)$. Then $[f]$ and $[g]$, the homotopy classes of maps from X to $\mathscr{U}(2)$ which are homotopic to f and g respectively, are elements of $[X, \mathscr{U}(2)]$. Suppose $[X, \mathscr{U}(2)]$ is abelian. Then the maps $f \cdot g(u, v) = u \cdot v$ and $g \cdot f(u, v) = v \cdot u$ are homotopic. We denote this by $f \cdot g \sim g \cdot f$. Since $\mu(u, v) = u \cdot v = f \cdot g(u, v)$ and $g \cdot f(u, v) = v \cdot u = \mu \circ \varphi(u, v)$, this is equivalent to saying $\mu \sim \mu \circ \varphi$. Let $\bar{\mu}$ and $\bar{\varphi}$ denote the restrictions of μ and φ, respectively, to $S\mathscr{U}(2) \times S\mathscr{U}(2)$. Then $\bar{\mu}: S\mathscr{U}(2) \times S\mathscr{U}(2) \to S\mathscr{U}(2)$ since $S\mathscr{U}(2)$ is a subgroup of $\mathscr{U}(2)$ and $\bar{\varphi}: S\mathscr{U}(2) \times S\mathscr{U}(2) \to S\mathscr{U}(2) \times S\mathscr{U}(2)$. Thus the above statements imply that $\bar{\mu} \sim \bar{\mu} \circ \bar{\varphi}$. Since $S\mathscr{U}(2)$ is homeomorphic to S^3, this is equivalent to saying that $\bar{\mu}$ is a homotopy commutative multiplication on S^3. This contradicts the fact that S^1 is the only sphere which admits a homotopy commutative multiplication [**2**, p. 192]. Hence $[f][g] \neq [g][f]$ and so $G(X, 2)/G_e(X, 2)$ is nonabelian.

If B and C are Banach algebras with groups H and K and identity components H_e and K_e, respectively, one can form their direct sum $A = B \oplus C$ by adding and multiplying componentwise and letting

$$\|(b, c)\| = \|b\| + \|c\| \quad \text{for } b \in B \text{ and } c \in C.$$

If G is the group of invertible elements of A and G_e is its identity component then $G \cong H \oplus K$ and $G_e \cong H_e \oplus K_e$ and so $G/G_e \cong H/H_e \oplus K/K_e$. Thus, one can obtain more complicated groups G/G_e by letting A be, for example, the algebra of all continuous functions from the disjoint union of a sphere S^k $(4 \leqq k \leqq 22)$ and $\mathscr{U}(2) \times \mathscr{U}(2)$ into M_n. Then A decomposes into the direct sum of algebras $A(S^k, n) \oplus A(\mathscr{U}(2) \times \mathscr{U}(2), n)$ and G/G_e is the direct sum of the corresponding factor groups of these algebras. If we let $n = 2$, this process yields an algebra whose factor group G/G_e is nonabelian and has torsion.

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