BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

BOOK REVIEWS

Lattice theory by Garrett Birkhoff. 3rd ed. New York: American Mathematical Society Colloquium Publications, 1967.

Lattice theory: *First concepts and distributive lattices* by George Grätzer, San Francisco; W. H. Freeman and Company, 1971.

In the past decade several books on the subject of lattice theory have appeared. These include works intended as reference books and those intended for use as course texts. Most of these books deal with one specific area in lattice theory, Maeda's *Theory of symmetric lattices* and von Neumann's classical *Continuous geometry* being but two of many examples.

Perhaps the best known book on lattices in general is Garrett Birkhoff's *Lattice theory*, first published in 1940, revised in 1948 and more recently in 1967. This is mainly a reference work—the field has grown too big to allow a complete and exhaustive treatment in one book. It is excellent reading, and the many references Birkhoff gives makes it the best place to start when one wishes to explore some portion of lattice theory or to appreciate the general flavor of the field.

Quite a different approach is used in George Grätzer's *Lattice theory*: *First concepts and distributive lattices*. This book is meant to be a text—it is written with the student in mind. Furthermore Grätzer limits himself to the study of distributive lattices—the book is geared in this direction from the beginning, and taken as a whole it gives a very well organized, cohesive treatment of this particular type of lattice.

Both authors have worked extensively in lattice theory. In particular both have made significant contributions to universal algebra. One has chosen to present an overall picture in a reference book; the other to write a specialized text. While the books differ in subject matter, and style, they agree in their high quality. There is, thus, no point in comparing or contrasting them, other than in making the remarks above—rather we will discuss each separately as two noteworthy examples of lattice theory books which differ in scope and intent.

R. P. Dilworth says in his review of the second edition of Birkhoff's *Lattice theory*, "... it should be the definitive work in the subject for some time to come, particularly, since the rapid growth of the field makes it unlikely that another such comprehensive account will be written." Subsequent events have proved Dilworth right—in this third edition Birkhoff claims to have made "no attempt at completeness" and wisely so; for there are several books listed in his bibliography which deal exclusively with the material covered in single chapters (or even sections) of *Lattice theory*.

If one had to label edition three with a single word, it seems that "modernization" rather than "revision" would be most suitable. The replacement of such cumbersome terms as "matroid lattice" with the more meaningful "geometric lattice" currently in usage makes the book much more readable, especially for students just entering the field. The reviewer for the second edition noted that between 1940 and 1948 "partly ordered set" had become "partially ordered set." One cannot resist remarking as a sign of our times that the term is now "poset."

It is natural that the largest single addition to the book is a chapter on universal algebra, where Birkhoff has made so many contributions. Aside from this, the various chapters have been reorganized, with developments made subsequent to the second edition included throughout the book—often in the form of exercises.

People working in lattice theory today tend to concentrate on special parts of the field—there are specialists in geometric lattices, orthmodular lattices, Noetherian lattices etc. It seems that this third edition is very useful as a reference book for lattice theorists who wish to know what sorts of things have been going on in branches of lattice theory other than their own.

Birkhoff's claim that the bibliography is incomplete is slightly misleading. While it is true that the list of papers he gives at the end of the book is short, there are many footnotes and references throughout the text. The reader who wishes to pursue some topic in more depth can easily get a starting point from these footnotes.

Birkhoff has organized his material into four main parts—general lattice-theoretic concepts, universal algebra, applications of lattice theory to various branches of mathematics, and mathematical structures which can be made into lattices.

He starts by presenting the basic ideas of lattice theory per se and discusses in the beginning chapters such topics as free, distributive and Boolean lattices, modularity and its applications to geometry, and completeness of lattices with the related ideas of closure operators and Galois connections. He next presents a survey of the results of universal algebra and approaches the applications of lattice theory to algebraic structures from this point of view. He presents among other things a generalized Jordan-Hölder theorem and discusses subgroup lattices with particular emphasis on solvable and supersolvable groups. Here he first introduces chain conditions which he uses extensively in later chapters.

From algebraic ideas Birkhoff moves into a treatment of lattice theory as it applies to various other fields of mathematics. He devotes a chapter to basic set-theoretic notions such as transfinite induction, ordinal arithmetic and the axiom of choice with its equivalent formulations. His discussion of the limiting process leads naturally to topological considerations. He presents some topological applications of lattice theory, in particular lattices of open sets of topological spaces, lattices of topologies of a space and the Stone Representation Theorem for Boolean algebras. He then discusses lattices which are also topological spaces and here introduces valuations and metric lattices. The ideas of topology and analysis are brought together in a chapter on Borel algebras and von Neumann lattices, and Birkhoff rounds out this portion of the book by giving a survey of logical and probabilistic applications of lattice theory. In all of these sections he omits many of the deeper or more complicated proofs. This is necessary due to limitations of space. However, the references and footnotes are especially useful here—the reader can easily find more detailed treatments of any of the topics Birkhoff presents.

The last five chapters of *Lattice theory* emphasize other mathematical structures which are also lattices such as lattice-ordered groups and rings, and vector lattices. Again Birkhoff develops an overall picture and directs the reader to other sources for details. As is the case throughout the book, he presents some unpublished results due both to himself and others—these also are carefully referenced.

There are one hundred and sixty-six "unsolved problems" in this third edition as compared to one hundred and eleven in the second and seventeen in the first! These range from the very specific such as "For which n do there exist plane projective geometries having $n + 1$ points on each line?" to the broad challenges such as "Develop a general theory of vector lattices over ordered fields." The problems vary considerably in difficulty—a few had already been solved before the book appeared, some others have been done since. The problems in this third edition are collected at the end of each chapter, which is an improvement over the previous arrangement.

The book seems too difficult for use as an introductory text; however it should be quite valuable to students working in the field. It does more than inform—it whets the appetite and arouses the reader's curiosity—what more can be said for a mathematics book!

In *Lattice theory: First concepts and distributive lattices*, Professor George Grätzer has written an excellent text. The need for a book dealing with distributive lattices in detail is best explained by Grätzer himself in his preface—"Historically, lattice theory started with (Boolean) distributive lattices; as a result, the theory of distributive lattices is the most extensive and most satisfying chapter in the history of lattice theory . . . .

Many conditions on lattices and on elements and ideals of lattices are weakened forms of distributivity. Therefore, a thorough knowledge of distributive lattices is indispensable for work in lattice theory . . .".

Grätzer's point is certainly well-taken. An obvious example is that of modularity—clearly a weakening of distributivity—and its own weakenings such as semimodularity, exchange properties etc., which are the cornerstones of continuous and combinatorial geometry. Certainly in the study of quantum and related logics (Boolean) distributive sublattices are of paramount importance in investigating the composition of these structures. It is hoped in fact that from the rich theory of distributive lattices one can eventually cull a more complete theory for various lattices which enjoy in one form or another a weakened version of distributivity.

Grätzer has divided his book into three chapters—first concepts, distributive lattices, and distributive lattices with pseudocomplementation—with the second and most important chapter comprising almost half of the book. Since the organization of the first chapter is quite interesting and unusual, a description of its content would be in order. First the definitions of lattice and poset are presented and there follows a very short but useful section on diagramming lattices. Next homomorphisms, ideals and the congruence lattice are described in some detail. As the book is geared in the direction of universal algebra, Grätzer includes almost immediately a section on lattice polynomials and identities, and presents distributivity from this point of view. Next he discusses free lattices and finally special elements such as 0, 1, complements and pseudocomplements. Presenting these topics in the above order is advantageous indeed. The student is not deluged immediately with a long list of definitions and trivial remarks, but is led very soon into an exploration of congruence lattices and free lattices instead of having to cover lots of introductory material first.

The presentation of the first chapter is clever from another point of view. The student has had a brief introduction to polynomials, congruences and freeness, so when he reaches the heart of the text in Chapter 2, he has the psychological advantage of meeting these ideas for the second time.

In the chapter on distributive lattices, Grätzer first gives some lattice-theoretic characterizations via the 5-element sublattices, and ideals. He presents the Birkhoff-Stone characterization via rings of sets and the Stone characterization of Boolean lattices via fields of sets. In a later section on topological representations he comes back to this last idea and presents the full Stone Representation Theorem.

Grätzer now specializes the ideas of freeness and congruence from the first chapter to the distributive case and brings in here the concept of generalized Boolean algebra. He next investigates Boolean algebras as generated by distributive lattices. Again freeness comes into play with a discussion of free distributive products and the chapter concludes with a study of the categories of Boolean and distributive lattices.

The third chapter on pseudocomplemented distributive lattices is more specialized and Grätzer here presents some very recent results of his as well as some due to Lee, Lakser, Balbes and others. He discusses Stone algebras, equational classes and some representation theorems.

As mentioned before the book is very well organized from a pedagogical point of view. It seems that in a course with this as text the student would see a unified theme along with relationships between old and new topics which is very important in making a course a success.

Two further features of this book are most attractive. At the end of each section there are many (20–50) exercises, and these exercises form an integral part of the text. Thus in large measure, the student learns by doing. The exercises vary a great deal in difficulty and the author indicates which few can be skipped without future damage in learning the material.

The other feature is this. At the end of each chapter the author includes a section on further topics and references which should give the gifted student a good start in making deeper independent investigations in this field. There are open problems included at the end of these sections—an excellent idea in a text to be used in the training of future mathematicians.

All in all the reviewer is quite favorably impressed with the book. The author remarks that he is working on a companion volume on general lattices. We eagerly await its appearance.

MARY KATHERINE BENNETT