BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

ON THE NONEXISTENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN NONREFLEXIVE SPACES[1]

BY ARRIGO CELLINA

Communicated by M. H. Protter, May 8, 1972

We consider the problem of the existence of solutions of ordinary differential equations in Banach spaces. Counterexamples in c_0 by Dieudonné [2] and in l_2 by Yorke [4] show that, in infinite-dimensional spaces, Peano's existence theorem need not necessarily be true. A natural question then is that of asking whether there could exist infinite-dimensional Banach spaces on which Peano's theorem holds, or otherwise, whether the truth of Peano's theorem is a characterization of the finite dimensionality.

This paper offers only a partial answer to this question: Our Theorem 1 below states that there exist no nonreflexive spaces on which Peano's theorem holds.

THEOREM 1. *Let X be a nonreflexive Banach space. Then there exists a continuous $F:R \times X \to X$ such that the Cauchy problem*

$$\text{(CP)} \qquad x' = F(t,x), \qquad x(0) = 0,$$

admits no solution on any nonvanishing interval $[a, b]$ containing the origin.

PROOF. Call B the unit ball of X. From R. C. James' characterization of reflexivity [3] it follows that there exists a $v \in X^*$, $\|v\| = 1$, such that, for every $x \in B$, $\langle v, x\rangle < 1$. We shall now construct a fixed-point-free continuous mapping f of B into itself with some special properties.

By definition of norm of v, we can recursively define a sequence of points x_n, $\|x_n\| = 1$, such that $\langle v, x_n\rangle < \langle v, x_{n+1}\rangle$ and $\langle v, x_n\rangle \to 1$, and consider the sets

$$O_1 = \{x \in B: \langle v, x\rangle < 2\langle v, x_2\rangle - 1\},$$

$$O_n = \{x \in B: 2\langle v, x_{n-1}\rangle - 1 < \langle v, x\rangle < 2\langle v, x_{n+1}\rangle - 1\}.$$

Then every O_j is open (relative to B) and their union covers B. Moreover it is not difficult to check that every point $x \in B$ belongs to at most two O_j and has a neighborhood that meets at most three O_j, so that the covering $\{O_n\}$ is locally finite and has a partition of unity subordinated

AMS 1970 *subject classifications.* Primary 34A10, 34G05; Secondary 46B10.
Key words and phrases. Ordinary differential equations, Cauchy problem, nonexistence of solutions, nonreflexive spaces.

[1] Supported in part by the CNR, Comitato per la Matematica.

to it. Call p_n this partition and define f to be

$$f(x) = \frac{1}{2}\sum_{i=1}^{\infty} p_i(x)(\langle v, x_{i+1}\rangle)^{-1}(\langle v, x\rangle + 1)x_{i+1}.$$

Then f is a continuous map $B \to X$. Moreover fix $x \in B$ and let $\hat{n}$ be such that x belongs at most to $O_{\hat{n}}$ and $O_{\hat{n}+1}$. Then

$$\begin{aligned}\|f(x)\| &\leqq \tfrac{1}{2}\{p_{\hat{n}}(x)(\langle v, x_{\hat{n}+1}\rangle)^{-1} + p_{\hat{n}+1}(x)(\langle v, x_{\hat{n}+2}\rangle)^{-1}\}(\langle v, x\rangle + 1)\\ &< \tfrac{1}{2}\{p_{\hat{n}}(x)(\langle v, x_{\hat{n}+1}\rangle)^{-1}\, 2\langle v, x_{\hat{n}+1}\rangle\} + \tfrac{1}{2}\{p_{\hat{n}+1}(x)(\langle v, x_{\hat{n}+2}\rangle)^{-1}\\ &\qquad \cdot 2\langle v, x_{\hat{n}+2}\rangle\}\\ &= 1.\end{aligned}$$

Therefore $f: B \to B$. In addition we have that

$$\langle v, f(x)\rangle = \frac{1}{2}\sum_{i=1}^{\infty} p_i(x)(\langle v, x\rangle + 1) = \tfrac{1}{2}(\langle v, x\rangle + 1).$$

This last equation implies the nonexistence of fixed points of f. In fact if $f(\xi) = \xi$, we would have $\langle v, \xi\rangle = 1$, contradicting our choice of v.

Let $\mathscr{F}: X \to B$ be an extension of f to the whole X, with range in B, and define the function $F: R \times X \to X$ by

$$\begin{aligned} F(t, x) &= 2t\mathscr{F}(x/t^2), && t \neq 0,\\ &= 0, && t = 0.\end{aligned}$$

Since $\|\mathscr{F}\| \leqq 1$, it follows that F is continuous on $R \times X$. Consider the Cauchy problem (CP) with the above defined F and let $y: [a, b] \to X$ be a solution, where $0 \in [a, b]$. For any given $t \in [a, b]$,

$$\begin{aligned}\|y(t)\| &\leqq \left|\int_0^t \|F(s, y(s))\|\, ds\right|\\ &\leqq \left|\int_0^t 2s\, ds\right| = t^2\end{aligned}$$

so that $\|y/t^2\| \leqq 1$.

Hence, along such a solution, $F(t, y) = 2tf(y/t^2)$. Moreover we have

$$\begin{aligned}\langle v, y\rangle' &= \langle v, y'\rangle = 2t\langle v, f(y/t^2)\rangle\\ &= 2t(1/2)(\langle v, y/t^2\rangle + 1) = (1/t)\langle v, y\rangle + t.\end{aligned}$$

The only solution of $\xi' = (1/t)\xi + t$, $\xi(0) = 0$, satisfying $|\xi| \leqq t^2$ is $\xi(t) = t^2$, so that $\langle v, y(t)\rangle = t^2$ for $t \in [a, b]$ or $\langle v, y(t)/t^2\rangle = 1$. Since $\|y(t)/t^2\| \leqq 1$, this contradicts our assumptions on v.

The technique used in the proof of the above theorem, of taking a special fixed-point-free self mapping f of B, extending it to $\mathscr{F}$ and defining $F(t, x)$ as $2t\mathscr{F}(x/t^2)$ can be used to give explicit examples of differential equations without existence in l_1 and l_∞, which are noteworthy because of their simplicity.

In l_∞ consider the mapping f given by

$$f(x_1, x_2, \ldots, x_n, \ldots) = (1 - \|x\|, |x_1|^{1/2}, \ldots, |x_{n-1}|^{1/2}, \ldots).$$

This is a continuous fixed-point-free mapping $B \to B$. Extend it to a $\mathscr{F}: X \to B$ and define F as before. It follows again that for a possible solution $y(t)$ we have $\|y(t)/t^2\| \leqq 1$ so that the following system has to hold:

$$y_1' = 2t(1 - \|y/t^2\|), \tag{1}$$

$$y_{n+1}' = 2|y_n|^{1/2}, \qquad n = 1, 2, \ldots. \tag{2}$$

We see that every y_i is nonnegative and that y_1 cannot be identically zero on any interval $[0, \delta]$ since then, on one hand, every y_n would be identically zero on that interval, while on the other, for every t, $\|y\| = \sup\{y_1(t), \ldots, y_n(t), \ldots\}$ should be t^2. So $0 = \inf\{t > 0 : y_1(t) > 0\}$.

Equations (2) are the equations of the process of successive approximations for the problem $x' = 2|x|^{1/2}$, $x(0) = 0$, with first approximation y_1. By a result of Dieudonné described in [**1**, p. 427], this process converges to the solution $x(t) = t^2$. Then on $[0, \delta]$, $\|y\| = t^2$ or $y_1' \equiv 0$. This implies $0 \equiv y_1 \equiv y_2 \equiv \ldots \equiv y_n \equiv \ldots$ on $[0, \delta]$ contradicting $\|y\| = t^2$.

In l_1 consider the mapping f defined by

$$f(x_1, x_2, \ldots, x_n, \ldots) = (1 - \|x\|, x_1, \ldots, x_{n-1}, \ldots).$$

It can be checked that this is a continuous fixed-point-free mapping of B into itself. Extending f and defining a continuous $F(t, x)$ as before, we see again that a possible solution y of (CP) has to be such that $\|y(t)\| \leqq t^2$ and has to satisfy, on any interval $[0, \delta]$, the system

$$y_1' = 2t(1 - \|y/t^2\|),$$

$$y_{n+1}' = 2y_n/t, \qquad n = 1, 2, \ldots.$$

Then every y_i' and every y_i is nonnegative. Moreover we have

$$\begin{aligned}\|y(t)\| &= \sum_{i=1}^{\infty} y_i(t) = \sum_{i=1}^{\infty} \int_0^t y_i'(s)\, ds \\ &= \int_0^t 2s(1 - \|y\|/s^2)\, ds + 2\sum_{i=1}^{\infty} \int_0^t y_i(s)/s\, ds.\end{aligned}$$

Since the sequence $g_n(s) = \sum_{i=1}^{n} y_i(s)/s$ is nondecreasing and bounded above by the integrable function $\|y(s)/s\|$,

$$\sum_{i=1}^{\infty} \int_0^t y_i(s)/s \, ds = \lim_{n\to\infty} \int_0^t g_n(s)\, ds = \int_0^t \|y(s)/s\| \, ds.$$

Hence from (3) we have, for every $t \in [0, \delta]$,

$$\|y(t)\| = \int_0^t 2s(1 - \|y/s\|) \quad ds + \int_0^t 2\|y/s\| \quad ds$$

$$= \int_0^t 2s \quad ds = t^2.$$

This implies $y_1' \equiv 0$ and consequently $0 \equiv y_1 \equiv y_2 \equiv \ldots \equiv y_n \equiv \ldots$. This contradicts $\| y \| = t^2$. Therefore no such solution y can exist.

REFERENCES

1. F. Brauer and S. Sternberg, *Local uniqueness, existence in the large, and the convergence of successive approximations*, Amer. J. Math. **80** (1958), 421–430. MR **20** #1806.

2. J. Dieudonné, *Deux exemples singuliérs d'équations différentielles*, Acta Sci. Math. Szeged **12** (1950), Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, pars B, 38–40. MR **11**, 729.

3. R. C. James, *Characterizations of reflexivity*, Studia Math. **23** (1963/64), 205–216. MR **30** #431.

4. J. Yorke, *A continuous differential equation in Hilbert space without existence*, Funkcial. Ekvac. **13** (1970), 19–21. MR **41** #8792.

ISTITUTO DI MATEMATICA, VIALE MORGAGNI 67/A, I 50134 FIRENZE, ITALY