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SUBMANIFOLDS, GROUP ACTIONS AND KNOTS. II

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§I contains typical examples of the application of the methods of [**3**], [**4**] to group actions. We compute equivariant knot cobordism for cyclic group fixed-point free actions. The surgery theory with coefficients that we need is outlined in §II. New algebraic K-theory functors $\Gamma_n(\Lambda' \to \Lambda)$ are introduced to solve geometric problems. For $\Lambda' = \Lambda$ these are Wall surgery groups [**7**].

The knot cobordism group of a manifold, or of a 2-plane bundle over a manifold is defined in §III and computed in general terms of the Γ-groups. In various cases, these Γ-groups are explicitly computed. The coefficient groups in this theory are isomorphic to the high dimensional knot-cobordism groups. The knot cobordism group of a manifold can be used to decide when sufficiently close codimension two embeddings differ, up to concordance, by a knot.

I. THEOREM 1. *Let T be a fixed-point free p.l. homeomorphism of the sphere Σ^{2k}, $k \geqq 3$, with $T^2 = 1$. Then there is at most one equivariant concordance class of invariant spheres of dimension $2k - 2$ in Σ.*

Santiago Lopez de Medrano [**5**] has computed which (Σ^{2k}, T) admit at least one invariant codimension two homotopy sphere; for example, this is always the case for k even.

THEOREM 2. *Let T be a fixed-point free p.l. homeomorphism of the sphere Σ^{2k+1} with $T^p = 1$, p odd. Then every fixed-point free Z_p action (S^{2k-1}, T') with S^{2k-1}/T' normally cobordant to the desuspension of Σ/T occurs, and only these occur, as the induced Z_p-actions on invariant spheres in codimension 2. If S^{2j+1} is a characteristic [**5**] invariant sphere of (Σ, T), there is a sequence of T-invariant spheres $S^{2j+1} \subset S^{2j+3} \subset \cdots \subset S^{2k-1} \subset S^{2k+1}$.*

Combining the above with results of Browder, Petrie and Wall [**1**], [**8**], it follows that actions induced on the invariant spheres in codimension 2 of (Σ, T), p odd, are in 1-to-1 correspondence with the elements of $Z \oplus Z \oplus \cdots \oplus Z = Z^{(p-1)/2}$.

Let T be a free action on the sphere Σ^{2k-1}, $k > 2$, with $T^n = 1$. Let $K(\Sigma, T)$ be the embeddings of (Σ, T), as an invariant subspace of a sphere

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S of dimension $2k+1$ equipped with a free action T' which restricts to T on Σ and with S/T' in a given normal cobordism class, classified up to equivariant concordance.

THEOREM 3. *There are "exact sequences" (i.e., free group actions on* $K(\Sigma, T)$ *with indicated quotient sets)*

$$0 \to \tilde{\Gamma}^s_{2k+2}(Z \to Z_n) \to K(\Sigma, T) \to Z \to 0, \qquad n \text{ odd or } k \text{ even},$$

$$0 \to \tilde{\Gamma}^s_{2k+2}(Z \to Z_n) \to K(\Sigma, T) \to Z_2 \to 0, \qquad n \text{ odd}, k \text{ odd},$$

$$0 \to \tilde{\Gamma}^s_{2k+2}(Z \to Z_n) \to K(\Sigma, T) \to 0, \qquad n \text{ even and } k \text{ odd}.$$

Here $\Gamma_{2k}(Z \to Z_n)$ denotes a reduced Grothendieck group of Hermitian forms defined over $Z[Z]$ and nonsingular over $Z[Z_n]$. Using a transfer homomorphism, we can use Theorem 9 to algebraically describe which knot cobordism classes contain a knot invariant under a free Z_n-action. More computations, including that of "fixed under a semifree action" knot cobordism will appear [**4**].

II. Let $f: \Lambda \to \Lambda'$ be an involution preserving homomorphism of rings equipped with involutions. In [**4**] we describe and in some cases compute a group $\Gamma_{2k}(\Lambda \to \Lambda')$ of Hermitian $(-1)^k$-symmetric forms over Λ which are nonsingular over Λ'. The map f is said to be locally epic if, given $u_1, \cdots, u_k$ in Λ', there is a unit λ in Λ' with $\lambda u_1, \lambda u_2, \cdots, \lambda u_k$ in the image f. Many of the results of this section work if f is only locally epic but, for simplicity, we assume f is actually onto. If f is an isomorphism, the Γ group is a surgery obstruction group as defined by C.T.C. Wall [**8**]. In what follows, homology is always taken with respect to local coefficients.

THEOREM 4. *Let* $(Y, \partial Y)$ *be a manifold pair of dimension* n *and* $f: Z[\pi_1 Y] = \Lambda \to \Lambda'$ *an onto map. A normal degree one map of manifold pairs* $g: (X, \partial X) \to (Y, \partial Y)$ *with* $\partial X \to \partial Y$ *a homology equivalence with coefficients in* Λ' *determines an element* $\sigma(g)$ *of* $\Gamma_{2k}(\Lambda \to \Lambda')$ *for* $n = 2k$ *and* $\sigma(g)$ *in* $L_{2k+1}(\Lambda')$ *for* $n = 2k+1$. *For* $n \geqq 6$ *the map* g *is normally cobordant to a* Λ' *homology equivalence, by a normal cobordism fixed on the boundary, if and only if* $\sigma(f) = 0$.

There is also a realization theorem for these obstructions. Note that if g is actually a homotopy equivalence on $\partial X \to \partial Y$, $\sigma(g)$ is in the image of the canonical homomorphism of $L_n(\Lambda)$ to $\Gamma_{2k}(\Lambda \to \Lambda')$ for $n = 2k$ and in the image of $L_n(\Lambda)$ in $L_n(\Lambda')$ for n odd. By first studying geometrically the case in which $\pi_1(\partial Y) = \pi_1(Y)$, we in [**4**] obtain obstruction groups, Γ, for the relative homology-equivalence problem in all dimensions greater than 5. Consequently, there is a surgery theory and a classification theory of homology-equivalent manifolds and submanifolds which has all the analogous formal properties and extends the surgery theory of C.T.C. Wall [**8**].

III. Let $C(M)$ denote equivalence classes of concordant embeddings, which are homotopic to the usual one of M in $M \times D^2$. More generally, for a 2-disc bundle ξ over M, $C(M, \xi)$ denotes equivalence classes of concordant embeddings of M in $E(\xi)$, the total space of ξ, which are homotopic to the 0 cross-section. The study of $C(M)$ and $C(M, \xi)$ plays a key role in determining when two embeddings, and, in particular, two "close" embeddings of M^n in W^{n+2} differ up to concordance by a "small knot." Given two embeddings $g, f: M \to E(\xi)$, we define $g + f$ by "thickening" g to an embedding $\bar{g}: E(\xi) \to E(\xi)$ and setting $g + f = \bar{g}f$. G_n denotes the knot cobordism group in dimension n.

THEOREM 5. *If M is a manifold of dimension $n \geqq 3$, $C(M, \xi)$ is an abelian group. $C(S^n) = G_n, n \geqq 3$. $C(S^n)$ is a direct summand of $C(M)$. An element of $C(M, \xi)$ is concordant to the connected sum of the usual inclusion g with a knot if and only if it is in the image of the homomorphism $C(S^n) \to C(M, \xi)$.*

Set $\Lambda = Z[\pi_1(\partial E(\xi))]$ and $\Lambda' = Z(\pi_1 M]$ and write $\psi: \Lambda \to \Lambda'$ for the map induced by $\partial E(\xi) \to M$. Let φ be the diagram

$$\begin{pmatrix} \Lambda \\ \downarrow \\ \Lambda \end{pmatrix} \begin{matrix} \to \\ \\ \xrightarrow{\psi} \end{matrix} \begin{pmatrix} \Lambda \\ \psi\downarrow \\ \Lambda' \end{pmatrix}$$

THEOREM 6. *For M^n a closed manifold, $n \geqq 4$, the following is an exact sequence*:

$$0 \to C(M, \xi) \to \Gamma^s_{n+3}(\varphi) \to L^s_{n+3}(\psi)/\text{image } [\Sigma M; G/PL].$$

To compute $\Gamma^s_{n+3}(\varphi)$ we employ the exact sequence $\ldots \to L^s_{n+3}(\Lambda) \to \Gamma^s_{n+3}(\psi) \to \Gamma^s_{n+3}(\varphi) \to \partial L^s_{n+2}(\Lambda) \to \ldots$. For n even, $\Gamma^s_{n+3}(\varphi)$ is caught in an exact sequence between two Wall surgery groups.

THEOREM 7. *If M is a simply-connected manifold of dimension $n \geqq 4$, then $C(M) = C(S^n)$. In particular, every inclusion M in $M \times D^2$ is, for n even, concordant to the usual inclusion and, for n odd, is concordant to the connected sum of the usual inclusion and a knot.*

Let φ_0 be the diagram induced from the group homomorphism $Z \to e$ of the integers to the trivial group.

$$\begin{pmatrix} Z[Z] \\ \downarrow \\ Z[Z] \end{pmatrix} \begin{matrix} \to \\ \\ \to \end{matrix} \begin{pmatrix} Z[Z] \\ \downarrow \\ Z[e] \end{pmatrix}$$

THEOREM 8. *The knot cobordism group G_n is isomorphic to $\Gamma_{n+3}(\varphi_0)$, $n \geqq 4$.*

THEOREM 9. *Denote the n-torus $S^1 \times S^1 \times \cdots \times S^1$ by T^n. Then $C(T^n) = C(S^n) \oplus [\Sigma T^n; G/PL]$, $n \geqq 4$.*

Theorem 9 is proved by showing that a formula generalizing to Γ-groups, the result of [7] and [8] on $L(Z \times G)$, is valid in precisely half the dimensions. The key geometric part of this argument for Γ-groups is based upon the technique employed in [2] in the proof of a splitting theorem for manifolds.

THEOREM 10. *Let M^n be a simply-connected manifold of dimension $n \geqq 4$. Then $C(S^n) \to C(M, \xi)$ is onto.*

THEOREM 11. *For M a simply-connected manifold, let $f: M^n \to W^{n+2}$ be an embedding with trivial normal bundle, $n \geqq 5$. If $g: M \to W$ is an embedding which is sufficiently close in the C^0 sense to f, then after composing g with a p.l. homeomorphism of M, the result is concordant to f for n even, and f connected sum with a knot for n odd.*

The extent to which this fails for $M = T^n$ is measured by Theorem 9. This contrasts sharply with the classical case of C^1 close embeddings.

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