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SUBMANIFOLDS, GROUP ACTIONS AND KNOTS. I

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This note and [4] outline new methods of classifying submanifolds of a manifold, submanifolds invariant under a group action, and submanifolds fixed under a group action. These methods solve many previously difficult problems associated with codimension two. In particular, they lead to a better understanding of the role of knot theory in the general placement problem for manifolds; this will be accomplished via the definition and computation of the local knot cobordism group of a manifold. Many of the results are most efficiently described in terms of new algebraic K-theoretic groups introduced in [4], [5].

§I has examples of the results on classification of embeddings of an n-dimensional manifold M^n in W^{n+2}. This is used to solve the problem of finding a purely geometric interpretation of the periodicity of knot cobordism [7], [8], [9] and [3]. The knot cobordism groups were introduced by Milnor and Fox in the classical case [6] and computed by Kervaire and Levine in the high-dimensional case. Our methods are basically independent of theirs.

§II contains an outline of codimension 2 surgery. The obstruction groups are very large in even dimensions. Some applications to group-actions, including problems of extending free cyclic group actions and the calculation of equivariant knot cobordism are in [4], [5]. Results of this type follow from the classification theory for homology equivalent manifolds developed there.

The connection between codimension 2 problems and homology equivalent manifolds has been suggested in previous work of the authors [3] and in Santiago Lopez de Medrano [10]. A detailed exposition of this theory, which involves our systematic generalization of the nonsimply connected surgery theory and surgery groups of C.T.C. Wall is, in [5]. In [4], [5], the knot cobordism group of a manifold, or of a 2-plane bundle over a manifold, is defined and computed in terms of an algebraic K-theory.

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Details, proofs and applications are in [5]. The results below are stated for piecewise-linear manifolds, group actions and locally-flat embedding. Analogous results are true for the differentiable and topological case,

I. Recall that two embeddings $f_i: X \to Y$, $i = 0, 1$, of manifolds are said to be concordant if there is an embedding $F: X \times I \to Y \times I$ with $F(i, x) = f_i(x)$, $i = 0, 1$. G_n denotes knot cobordism group in dimension n, i.e., the group of concordance classes of embeddings of the n-sphere S^n in S^{n+2}. Kervaire showed that $G_{2k} = 0$. Levine showed G_{2k+1} was an infinite direct sum of copies of Z, Z_2 and Z_4 and observed an algebraic isomorphism $G_k \cong G_{k+4}$, $k \neq 1, 3$. In [3], it was proved that, for topological embeddings, $G_k^{\text{top}} \cong G_{k+4}^{\text{top}}$, $k \geqq 3$. By obtaining below two different classifications of the concordance classes of embeddings of $S^n \times M$ in $S^{n+2} \times M$, the problem of giving a geometric interpretation to the periodicity of knot cobordism is solved,

Let i denote the usual inclusion of S^n in S^{n+2} and, for a closed manifold M of dimension k, let $j = i \times 1_M: S^n \times M \to S^{n+2} \times M$. Two embeddings α, β of $S^n \times M$ in $S^{n+2} \times M$ are said to be equivalent if there are p.l. homeomorphisms $\rho_1: S^n \times M \to S^n \times M$, $\rho_2: S^{n+2} \times M \to S^{n+2} \times M$ with ρ_i, $i = 1, 2$, commuting up to homotopy with the projection onto M, with $\beta = \rho_2 \alpha \rho_1$. $G_n(M)$ will denote the concordance classes of equivalent embeddings of $S^n \times M$ in $S^{n+2} \times M$ which are homotopic to j. A map $\varphi_M^n: G_{n+k} \to G_n(M)$ is defined by letting, for a knot α representing an element of G_{n+k}, $\varphi_M^n(\alpha)$ be the knot arithmetic sum of j and α.

THEOREM 1. *Let M be a closed simply-connected manifold of dimension $k > 3$. Then $\varphi_M^n: G_{n+k} \to G_n(M)$, $n > 1$, is a one-to-one correspondence.*

A map $P_M^n: G_n \to G_n(M)$ is defined for a knot α by taking its product with M.

THEOREM 2. *Let M be a closed simply-connected manifold of dimension $k = 4q$ with index ± 1. Then $P_M^n: G_n \to G_n(M)$, $n > 3$, is a one-to-one correspondence.*

The desired geometric periodicity is now obtained by taking M to be CP^2.

THEOREM 3. *$(\varphi_{CP^2}^n)^{-1} P_{CP^2}^n: G_n \to G_{n+4}$, $n > 3$, is an isomorphism of groups.*

If $n = 3$, $(\varphi_{CP^2}^n)^{-1} P_{CP^2}^n$ is injective with cokernel Z_2.

(For topological knots the corresponding map is an isomorphism [3].) The map P_M^n is still injective if the index of M is odd. Bredon [1] has a different geometric description of the periodicity map $G_n \to G_{n+4}$.

II. For odd-dimensional manifolds, we show that the ambient surgery obstruction in codimension two is the abstract surgery obstruction. Precisely, let $f: W \to V$ be a homotopy equivalence of closed manifolds of dimension n. A submanifold M of V determines, by making f transverse to M, an induced surgery problem and hence, for M a codimension 2 submanifold, an element $\sigma_M(f)$ of the Wall surgery obstruction group $L_{n-2}(\pi_1 M)$.

THEOREM 4. *If $n = 2k + 1$, $k > 2$, the map f is homotopic to a map (which we continue to call f) transverse regular to M with $f^{-1}(M) \to M$ a homotopy equivalence if and only if $\sigma_M(f) = 0$. Moreover, if $\sigma_M(f) = 0$, among the manifolds homotopy equivalent to M, those in one normal cobordism class, and only those, will occur as $f^{-1}M$ for some f in the given homotopy class.*

There is a corresponding relative form of the above result for manifolds with boundary. Recalling that $L_{2k-1}(0) = 0$, a special case is the following.

THEOREM 5. *Let $f: W \to V$ be a homotopy equivalence of closed manifolds of dimension $2k + 1$, $k > 2$. Let M^{2k-1} be a simply-connected submanifold of V. Then f is homotopic to a map, transverse regular to M (and which we continue to call f) with $f^{-1}(M) \to M$ a homotopy equivalence. Moreover, $f^{-1}M$ is uniquely determined by this.*

Using the relative form of this theorem for $M = S^{2k} \times I$ and $V = S^{2k+2} \times I$, we obtain the classical result of Kervaire on the vanishing of the even-dimensional knot cobordism groups.

The functors Γ introduced in [4], [5] describe the obstructions to ambient codimension 2 surgery in even-dimensional manifolds. L denotes the Wall surgery group functor [11]. Given M^{2k-2} a submanifold of W^{2k}, there is a homomorphism

$$\rho: L_{2k-1}(\pi_1 M) \to \operatorname{Ker}(\Gamma_{2k}(Z[\pi_1(W - M)] \to Z[\pi_1 W]) \to L_{2k}(\pi_1 W)).$$

THEOREM 6. *Let $(M^{2k-2}, \partial M)$ be a proper submanifold of $(V^{2k}, \partial V)$ and let $f:(W, \partial W) \to (V, \partial V)$ be a homotopy equivalence of manifolds restricting to a homotopy equivalence $\partial W \to \partial V$. Assume, moreover, that $f^{-1}(\partial M) \to \partial M$ is a homotopy equivalence. Then f is homotopic by a homotopy which is fixed on ∂W to a map (which we continue to call f) with $f^{-1}(M) \to M$ a homotopy equivalence if and only if $\sigma_M(f) = 0$ and an obstruction $\tau(f)$, defined if $\sigma_M(f) = 0$, as an element of the cokernel of ρ, vanishes.*

The proof of Theorem 4 uses the cobordism extension technique introduced by Browder to study embeddings in codimension greater than 2

[2] and new methods of studying homology equivalent odd-dimensional manifolds.

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