BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF A SEQUENCE OF ITERATES FOR QUASI-NONEXPANSIVE MAPPINGS

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Communicated by Eugene Isaacson, May 1, 1972

Introduction. The purpose of this note is to present two theorems which provide necessary and sufficient conditions for the convergence of the successive approximation method (Theorem 1) and of the convex combination iteration method (Theorem 2) for quasi-nonexpansive mappings defined on suitable subsets of Banach spaces and with nonempty sets of fixed points. We also indicate briefly how these theorems are used to deduce a number of known, as well as some new, convergence results for various special classes of mappings of nonexpansive, P-compact, and 1-set-contractive type which recently have been extensively studied by a number of authors. Complete proofs and detailed discussion of the results presented in this note will be given in [**15**].

1. Let X be a real Banach space, D a closed subset of X, and T a continuous mapping of D into X such that T has a nonempty set of fixed points $F(T) \subset D$ and

(1) $\|T(x) - p\| \leqq \|x - p\|$ for all x in D and p in $F(T)$.

In what follows we shall refer to T satisfying the above conditions as *quasi-nonexpansive*. Condition (1) was introduced by Tricomi [**17**] for real functions and further studied by Diaz and Metcalf [**3**] and Dotson [**4**] for mappings in Banach spaces. It is not hard to see that the class of quasi-nonexpansive mappings properly includes the class of nonexpansive maps (i.e., T is such that $\|Tx - Ty\| \leqq \|x - y\|$ for $x, y \in D$) with $F(T) \neq \emptyset$.

The first basic result of this note is the following new theorem which characterizes the convergence of Picard iterates for quasi-nonexpansive maps.

THEOREM 1. *Let X be a real Banach space, D a closed subset of X, and T a quasi-nonexpansive mapping of D into X. Suppose there exists a*

AMS 1970 *subject classifications*. Primary 47H15; Secondary 47H05.

Key words and phrases. Iteration methods, convergence, quasi-nonexpansive mapping, fixed points, nonexpansive mappings, condensing and 1-set-contractive mappings, P-compact mappings.

[1] Partially supported by the NSF grant GP-20228.

point x_0 *in* D *such that the sequence* $\{x_n\}$ *of iterates lies in* D, *where* x_n $(= T^n(x_0))$ *is given by*

$$x_n = T(x_{n-1}), \qquad n = 1, 2, 3, 4, \cdots. \tag{M1}$$

Then $\{x_n\}$ *converges to a fixed point of* T *in* D *if and only if the following condition* (C) *holds*:

$$\lim_n d(x_n, F(T)) = 0, \tag{C}$$

where $d(x_n, F(T))$ *denotes the distance from* x_n *to* $F(T)$.

To outline the proof of Theorem 1 we first note that if $\{x_n\}$ converges to a fixed point of T in D, the condition (C) clearly holds. Conversely, if condition (C) holds, then in view of (1) it is not hard to show that $\{x_n\}$ is a Cauchy sequence whose limit lies in $F(T)$.

Although, the principal practical usefulness of Theorem 1 lies in the fact that it yields an analogous result for the convex combination iteration method (M2) below, nevertheless we mention here two of its practically useful corollaries which among others include the somewhat stronger versions of the corresponding convergence results in [2], [3].

PROPOSITION 1. *Suppose* X, D, T *and* x_0 *satisfy the conditions of Theorem* 1. *Suppose further that*

(2) T *is asymptotically regular at* x_0 *in the sense of* [2] (*i.e.*, $\|T^n(x_0) - T^{n+1}(x_0)\| \to 0$ *as* $n \to \infty$).

(3) *If* $\{y_n\}$ *is any sequence in* D *such that* $\|(I - T)(y_n)\| \to 0$ *as* $n \to \infty$, *then* $\lim_n \inf d(y_n, F(T)) = 0$.

Then $\{x_n\}$ *determined by the process* (M1) *converges to a fixed point of* T *in* D.

PROPOSITION 2. *Suppose* X, D, T, *and* x_0 *satisfy the conditions of Theorem* 1. *Suppose further that*

(4) *for every* x *in* $D - F(T)$, *there exists* p_x *in* $F(T)$ *such that*

$$\|T(x) - p_x\| < \|x - p_x\|.$$

(5) *The sequence* $\{x_n\}$ *determined by* (M1) *contains a convergent subsequence* $\{x_{n_j}\}$ *converging to some* x^* *in* D.

Then $x^* \in F(T)$ *and the entire sequence* $\{x_n\}$ *converges to* x^*.

We add in passing that Proposition 2 has been first obtained in [3] but under the condition

(4′) *for every* x *in* $D - F(T)$ *and every* p *in* $F(T)$,

$$\|T(x) - p\| < \|x - p\|,$$

which is more stringent than our condition (4). In fact, an example is given in [15] for which condition (4) holds but not (4′).

In view of Theorem 1, to prove Propositions 1 and 2, it suffices to show that conditions (2) and (3) as well as (4) and (5) imply the condition (C).

REMARK 1. Although we have formulated Theorem 1 and Propositions 1 and 2 in terms of Banach spaces, a careful examination of their proofs shows that only the distance function between points and sets has been used. Consequently, the above results are also valid for complete metric spaces.

It is known (see, for example, [9]) that even if D is the unit ball $B(0, 1)$ in a uniformly convex Banach space X and T is a nonexpansive compact mapping of B into B, then the method (M1) will not converge without some additional conditions. It follows from Theorem 1 that (C) is the weakest condition under which the method (M1) converges. Unfortunately, the verification of condition (C) for $\{x_n\}$ determined by (M1) can be obtained only under very restrictive conditions on T.

However, it was shown in [9] and then in [16] that if D is a bounded closed convex subset of a uniformly convex Banach space, T a nonexpansive compact map of D into D, and if instead of the ordinary method (M1) we consider the convex combination method

$$\text{(M2)} \quad x_n = T_\lambda(x_{n-1}), \qquad x_0 \in D, \quad T_\lambda = \lambda T + (1 - \lambda)I, \qquad \lambda \in (0, 1),$$

then $\{x_n\}$ converges to a fixed point of T in D. The results of [9], [16] were extended first in [12] and then in [2] to certain classes of noncompact nonexpansive maps. The further extensions of the results in [2] to quasi-nonexpansive mappings were given in [3], [4].

Using our Theorem 1, we prove an analogous result for the iteration method (M2) which not only unifies and extends the convergence results of [9], [16], [12], [2], [3], [4] but also provides a necessary and sufficient condition for the convergence of the method (M2).

THEOREM 2. *Let X be a Banach space, D a closed convex subset of X, and T a quasi-nonexpansive map of D into X. Suppose there exists a point x_0 in D such that, for some λ in* $(0, 1)$, *the sequence* $\{x_n\} = \{T_\lambda^n(x_0)\}$ *given by* (M2) *lies in D.*

Then $\{x_n\}$ converges to a fixed point of T in D if and only if

$$\text{(C}'\text{)} \qquad \lim_n d(T_\lambda^n(x_0), F(T)) = 0.$$

In view of the fixed point theorem of Browder-Kirk-Gohde [1], [8], [7] and the theorem on asymptotic regularity of Browder-Petryshyn [2], an immediate consequence of Theorem 2 is the following proposition which unites and extends the convergence results of [9], [16], [12], [2].

PROPOSITION 3. *Let X be a uniformly convex Banach space, D a closed bounded convex set in X, and T a nonexpansive map of D into D such that either*

(6) *$I - T$ maps closed sets in D into closed sets in X or*

(7) *T is demicompact at 0 in the sense of* [**12**].

Then, for any x_0 in D and λ in $(0, 1)$, the sequence $\{x_n\}$ determined by (M2) *converges to a fixed point of T in D.*

REMARK 2. Joram Lindenstrauss informed the first author that he had constructed an example of a nonexpansive mapping T of a unit ball $B(0, 1)$ of a Hilbert space into itself with $F(T) \neq \emptyset$ for which the sequence of iterates $\{T^n_{1/2}(x_0)\}$ does not converge to a fixed point of T in B. Consequently, for the sequence $\{x_n\}$ constructed by the simple method (M2) to be convergent to a fixed point of a nonexpansive mapping T of D into D with $F(T) \neq \emptyset$, some additional conditions on T have to be imposed. It appears that our condition (C′) is the weakest (since it is also necessary) condition which insures the convergence of the method (M2).

We add in passing that the iteration result of Edelstein [**5**] for a nonexpansive compact map in a strictly convex Banach space X as well as the result of Petryshyn [**13**] for set-condensing nonexpansive mappings in general Banach spaces are also deducible from Theorem 2. It is also shown that if $D = B(0, r) \subset X$, X is strictly convex, and $T: B \to X$ is a nonexpansive set-condensing or ball-condensing map which satisfies the Leray-Schauder condition on the boundary ∂B of B, then there exists an open set Q in B such that, for each x_0 in Q and λ in $(0, 1)$, the sequence $\{T^n_\lambda(x_0)\}$ lies in B and converges to a fixed point of T in B.

In view of the fixed point theorem of Petryshyn [**14**] for 1-set and 1-ball contractive mappings and the fixed point theorem of Frum-Ketkov [**6**] (with the proof of the latter due to Nussbaum [**10**]), Theorem 2 is also applicable to the classes of mappings studied in [**14**], [**6**], [**10**]. Mappings which are P-compact in the sense of [**11**] are also treated.

Finally, we add that the problem of the weak convergence of the sequence of iterates $\{x_n\}$ determined by (M1) or (M2), when (C) or (C′) does not hold, is also treated.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044. MR **32** #4564.

2. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 566–570. MR **32** #8155a.

3. J. B. Diaz and F. T. Metcalf, *On the set of subsequential limit points of successive approximations*, Trans. Amer. Math. Soc. **135** (1969), 459–485. MR **38** #2644.

4. W. G. Dotson, *On the Mann iterative process*, Trans. Amer. Math. Soc. **149** (1970), 65–73. MR **41** #2477.

5. M. Edelstein, *A remark on a theorem of M. A. Krasnosel'skiĭ*, Amer. Math. Monthly **73** (1966), 509–510. MR **33** #3072.

6. R. L. Frum-Ketkov, *Mappings into a Banach space sphere*, Dokl. Akad. Nauk SSSR **175** (1967), 1229–1231 = Soviet Math. Dokl. **8** (1967), 1004–1006. MR **36** #3181.

7. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258. MR **32** #8129.

8. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006. MR **32** #6436.

9. M. A. Krasnosel'skiĭ, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk **10** (1955), no. 1 (63), 123–127. (Russian) MR **16**, 833.

10. R. N. Nussbaum, *Some asymptotic fixed point theorems* (to appear).

11. W. V. Petryshyn, *On a fixed point theorem for nonlinear P-compact operators in Banach space*, Bull. Amer. Math. Soc. **72** (1966), 329–334. MR **33** #1768.

12. ———, *Construction of fixed points of demicompact mappings in Hilbert space*, J. Math. Anal. Appl. **14** (1966), 276–284. MR **33** #3147.

13. ———, *Structure of the fixed points sets of k-set-contractions*, Arch. Rational Mech. Anal. **40** (1970/71), 312–328. MR **42** #8358.

14. ———, *A new fixed point theorem and its application*, Bull. Amer. Math. Soc. **78** (1972), 225–229.

15. W. V. Petryshyn and T. E. Williamson, Jr., *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. (to appear).

16. H. Schaefer, *Über die Methode sukzessiver Approximationen*, Jber. Deutsch. Math. Verein **59** (1957), Abt. 1, 131–140. MR **18**, 811.

17. F. Tricomi, *Un teorema sulla convergenza delle successioni formate delle successivo iterate di una funzione di una variable reale*, Giorn. Mat. Battaglini **54** (1916), 1–9.

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