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AN EQUIVALENT FORMULATION OF THE INVARIANT SUBSPACE CONJECTURE

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The principal purpose of this announcement is to present an equivalent formulation of the invariant subspace conjecture for bounded linear operators acting on a Hilbert space H. Specifically, the conjecture asserts that if $B(H)$ denotes the algebra of bounded linear operators on H and $A \in B(H)$, then A has a nontrivial invariant subspace. We show that the conjecture can be reduced to the study of operators having the property that their invariant subspaces are reducing spaces. In our earlier announcement of this result we called such an operator "completely normal" (cf. [**2**]); however, since then we have been convinced (by P. R. Halmos) that "reductive" is a more appropriate term.

Throughout this note H will denote an arbitrary Hilbert space. An element $A \in B(H)$ is called *reductive* if, and only if, each invariant subspace of A reduces A. $A \in B(H)$ is called *normal-free* if there is no reducing subspace $\mathcal{M}$ for A other than (0) such that $A|_{\mathcal{M}}$ (the restriction of A to $\mathcal{M}$) is normal. If $S \subset B(H)$, then S' is the set of all $B \in B(H)$ such that $BA = AB$ and $BA^* = A^*B$ for every $A \in S$ where A^* denotes the adjoint of A. An element $C \in B(H)$ is called transitive if there are no invariant subspaces for C other than (0) and H.

Our basic result is that if $\dim H > 1$, then the invariant subspace conjecture is correct if, and only if, every reductive element of $B(H)$ is normal. Inasmuch as the proof of the result requires an elaborate use of direct integral theory for rings of operators, we have not given proofs to the theorems. The complete proofs are expected to appear in a forthcoming monograph on direct integral theory and its applications.

If $A \in B(H)$ and is reductive and $\mathcal{M}$ an invariant subspace for A then we define

$$N(\mathcal{M}) = \mathcal{M} \cap \{ A^{*n}(AA^* - A^*A)\xi \mid \xi \in \mathcal{M}, \quad n = 0, 1, \cdots \}^{\perp}.$$

In particular we set $H_0 = N(H)$ and let P_0 denote the projection of

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H onto H_0. The following theorem is one of the basic structure theorems for reductive operators.

THEOREM 1. *Let A be a reductive operator on the Hilbert space H.*

(a) *If $\mathscr{M}$ is an invariant subspace for A, then $N(\mathscr{M})$ is an invariant subspace for A which is contained in $\mathscr{M}$.*

(b) *If $\mathscr{M}$ and $\mathscr{M}_1$ are invariant subspaces of A with $\mathscr{M}_1 \subset \mathscr{M}$, then $A|_{\mathscr{M}_1}$ is normal if, and only if, $\mathscr{M}_1 \subset N(\mathscr{M})$.*

(c) *If $\mathscr{M}$ is an invariant subspace for A, then $A|_{\mathscr{M}}$ is normal if, and only if, $\mathscr{M} \subset H_0$. $A|_{\mathscr{M}}$ is normal-free if and only if $\mathscr{M} \subset H_0^{\perp}$.*

(d) *P_0 is a projection in the center of $\{A\}'$, If $\mathscr{M}$ is an invariant subspace for A, then $\mathscr{M}$ can be decomposed uniquely as an orthogonal direct sum $\mathscr{M}_1 + \mathscr{M}_2$ of invariant subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ for A such that $A|_{\mathscr{M}_1}$ is normal and $A|_{\mathscr{M}_2}$ is normal-free; moreover, $\mathscr{M}_1 = P_0(\mathscr{M})$ and $\mathscr{M}_2 = (I - P_0)(\mathscr{M})$.*

As an immediate consequence of Theorem 1, we have

COROLLARY 1.1. *Let A be a reductive operator acting on the Hilbert space H.*

(a) *If $\mathscr{M}$ is an invariant subspace for A which is contained in its normal free subspace, then either $\mathscr{M} = (0)$ or $\dim \mathscr{M}$ is infinite.*

(b) *If $\dim H < \infty$, then A is normal.*

REMARK. In view of Theorem 3 below, (b) of the above corollary allows us to assert that "if $1 < \dim H < \infty$ and $A \in B(H)$, then A has a nontrivial invariant space" without resorting to the Jordon form of A.

One of the difficulties encountered in the applications of direct integral theory is proving that vector valued functions are measurable. Our main tool in this direction is our next theorem which we call "the metric approximation theorem." In order to state the theorem we shall make a few measure theoretic conventions. If (M, Σ, μ) is a finite measure space, then μ^* denotes the outer measure corresponding to μ. A *measurable cover* of a subset T of M is a measurable set U ($\in \Sigma$) such that every measurable subset of $U - T$ is null and such that $T - U$ is a subset of a null set. (We do not assume that μ is complete.) Such a measurable cover U always exists and $\mu(U) = \mu^*(T)$. Moreover, if U_1 is a second measurable subset of M, then we observe that U_1 is a measurable cover of T if, and only if, the symmetric difference $U \triangle U_1$ is null. This observation leads to all of the properties of measurable covers that are needed for the next theorem.

THEOREM 2. *Let (M, Σ, μ) be a finite measure space, T a subset of M, $\{K_i, \delta_i^0\}_{i=1}^n$ a finite family of separable metric spaces (δ_i is the metric* *on the space* K_i), $F_i : M \to K_i$ *a function for* $i = 1, 2, \cdots, n$, $\varepsilon > 0$ *and* $0 < \alpha < 1$.

(a) *Then there exists a subset* T_1 *of* T *and measurable functions* G_i *with finite range for* $i = 1, 2, \cdots, m$ *such that* $\mu^*(T_1) \geqq \alpha\mu^*(T)$ *and such that* $\delta_i(F_i(m), G_i(m)) < \varepsilon$ *for* $i = 1, 2, \ldots, n$ *if* $m \in T_i$.

(b) *If, in addition to the above hypothesis, we assume that* M *is a topological space,* Σ *contains all clopen (i.e. open and closed) subsets of* M, *and every subset of* M *has a clopen measurable cover, then the functions* $\{G_i\}_{i=1}^n$ *in the conclusion above can be chosen to be continuous.*

In our applications of Theorem 2 we only use the case where $n = 2$.

The proofs of our main results require a great deal of the theory of direct integral decompositions for von Neumann algebras. The necessary literature for our use of the theory can be found in [**3**], [**5**], [**6**] and [**8**]. In what follows, E will denote a weakly closed symmetric subring of $B(H)$ such that its commutant E' has a cyclic vector for H. M denotes the maximal ideal space of E and for $B \in E$ the mapping $B \to \hat{B}(m)$ denotes the Gelfand transform of B.

THEOREM 3. *Suppose* $A \in E'$ *with direct integral decomposition* $A \cong \int_M^\oplus A_m \, d\mu(m)$ *and* T *denotes the set of* $m \in M$ *for which* A_m *is not transitive.*

(a) T *is measurable. Furthermore, if* $0 < \alpha < 1$, *there exists a clopen set* $U \subset M$ *with* $\mu(U) \geqq \alpha\mu(T)$ *and vectors* $\xi^i \in H$, $i = 1, 2$, *with direct integrals* $\int_M^\oplus \xi_m^i \, d\mu(m)$ *such that*

(i) $\|\xi_m^i\| = \chi_u(m)$ *everywhere,* $\chi_u(m)$ *the characteristic function of* U *on* M, *and*

(ii) *for every nonnegative integer* n, $(A_m^n \xi_m^1, \xi_m^2) = 0$ *a.e.*

(b) *If, in addition,* A *is reductive and* E *is a maximal symmetric commutative subring of* $\{A, A^*\}'$, *then* $\mu(T) = 0$. *In this case* M *is the disjoint union of two clopen sets* M_∞ *and* M_1 *with* $\dim(H_m) = 1$, $m \in M_1$, *and* $\dim(H_m) = \aleph_0$, $m \in M_\infty$; *moreover* P_∞ *and* P_1 *where* $\hat{P}_\infty(m) = \chi_{M_\infty}(m)$ $\hat{P}_1(m) = \chi_{M_1}(m)$ *are respectively the normal-free and normal projections of* A.

COROLLARY 3.1. *If* H *is a Hilbert space and* $\dim(H) > 1$, *then the following statements are equivalent.*

(i) *Every element of* $B(H)$ *has a nontrivial invariant subspace.*

(ii) *If* A *is a reductive element of* $B(H)$, *then* A *is normal.*

We note that other hypotheses will lead to the validity of (ii) of Corollary 3.1; for example, using the results of Aronszajn-Smith and Bernstein-Robinson (cf. [**1**], [**7**], or [**4**]) we have the following:

COROLLARY 3.2. *If* A *is a reductive element of* $B(H)$ *which is polynomially compact, then* A *is normal.*

Also, we have

COROLLARY 3.3 *If A is a reductive element of $B(H)$ and $A \cong \int_m^{\oplus} A_m \, d\mu(m)$, then A_m is transitive a.e.*

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