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# INFINITE-DIMENSIONAL METHODS IN FINITE-DIMENSIONAL GEOMETRIC TOPOLOGY

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1. **Introduction.**[2] We use the methods of infinite-dimensional topology to derive new information about the topology of euclidean spaces and manifolds. The idea is to partition euclidean $n$-space $E^n$ into a $k$-dimensional pseudo-boundary $(0 \leqq k < n)$ and an $(n - k - 1)$-dimensional pseudo-interior, and to deduce negligibility theorems analogous to those known for the pseudo-boundary and the pseudo-interior (denoted by $s$) of the Hilbert cube $I^\omega$. Since $s$ is homeomorphic to Hilbert space $l_2$, there is a sense in which we are giving the correct finite-dimensional analogues of $l_2$ (see §5).

DEFINITION. A subset $X$ of a metric space $Y$ (with metric $d$) is *strongly negligible in* $Y$ if, for each open set $U$ in $Y$ and each map $\varepsilon: U \to R^+$, there is a homeomorphism $h: Y \to Y - (X \cap U)$ fixing $Y - U$ such that $d(x, h(x)) < \varepsilon(x)$ for all $x \in U$. This is a topological property independent of $d$.

THEOREM 1.1. *$E^n$ is the union of two disjoint dense subsets $B^k$ and $P^{n-k-1}$ such that* (1) *if $n \leqq 2k + 1$, any $\sigma$-compact subset of $P^{n-k-1}$ is strongly negligible in $P^{n-k-1}$, and* (2) *if $n \geqq 2k + 1$, any compact subset of $B^k$ is strongly negligible in $B^k$. If $n = 2k + 1$, any $k$-dimensional compactum can be embedded in $B^k$ or in $P^k$.*

NOTATION. Superscripts on spaces, e.g., $B^k$, $P^{n-k-1}$, indicate dimension.

We call $B^k$ of Theorem 1.1 *the universal $k$-dimensional pseudo-boundary of $E^n$*. It is built out of Menger universal compacta [**13**], [**17**]. (See §3.) $P^{n-k-1}$ of Theorem 1.1 is the corresponding pseudo-interior.

Another kind of $k$-dimensional pseudo-boundary in $E^n$ can be built out of polyhedra as follows.

Let $J_0$ be a rectilinear PL triangulation of $E^n$, all $n$-simplexes having the same diameter. Let $J_i$ $(i \geqq 1)$ be the $i$th barycentric subdivision of $J_0$, its $k$-skeleton being $J_i^k$. The *polyhedral $k$-dimensional pseudo-boundary of $E^n$* is $\tilde{B}_n^k = \bigcup_{i=1}^\infty |J_i^k|$. The corresponding pseudo-interior is $\tilde{P}_n^{n-k-1} = E^n - \tilde{B}_n^k$.

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[2] This is a summary of a paper entitled *Pseudo-boundaries and pseudo-interiors in euclidean spaces and topological manifolds.*

TERMINOLOGY. A *polyhedron* is a space homeomorphic to the body of a countable locally finite simplicial complex; a *subpolyhedron of* $E^n$ is the body of such a complex when linearly embedded in $E^n$; a polyhedron $X$ in $E^n$ is *tame* if there is a homeomorphism $h$ of $E^n$ such that $h(X)$ is a subpolyhedron of $E^n$.

THEOREM 1.2. (1) *Any subpolyhedron of* $E^n$ *of dimension* $\leqq n-k-1$ *can be embedded in* $\tilde{P}_n^{n-k-1}$ *so as to be tame in* $E^n$; (2) *if* $n \leqq 2k+1$, $n \neq 4$, *then the countable union of polyhedra, each tame in* $E^n$ *and lying in* $\tilde{P}_n^{n-k-1}$, *is strongly negligible in* $\tilde{P}_n^{n-k-1}$; (3) *if* $n \geqq 2k+1$, $n \neq 4$, *then any compact subset of* $\tilde{B}_n^k$ *is strongly negligible in* $\tilde{B}_n^k$; (4) *if* $0 \leqq k \leqq n-3$, $k \neq 1$, $n \neq 4$, *then the countable union of compact polyhedra of dimension* $\leqq k$ *in* $\tilde{P}_n^{n-k-1}$ *is strongly negligible in* $\tilde{P}_n^{n-k-1}$; (5) *any* $k$-*dimensional polyhedron can be embedded in* $\tilde{B}_{2k+1}^k$ *or in* $\tilde{P}_{2k+1}^k$.

In all of these constructions the analogy with $I^\omega$ is striking. It is explained in §5.

The theory of negligibility is briefly set out in §2. It is then applied in §§3 and 4. The analogy with $I^\omega$ is discussed in §5. The extension to manifolds is in §6. Theorem 1.1 follows from the ideas of §3 and Theorem 1.2 from those of §4.

2. **Pseudo-boundaries in complete metric spaces.** Given a complete metric space $Y$, when can $Y$ be partitioned into a set $B$ (analogous to the pseudo-boundary of $I^\omega$) and a set $P$ (analogous to the pseudo-interior of $I^\omega$) so that "tame" subsets of $B$ or $P$ are negligible and what constitutes a family of "tame" sets? The answer, Theorems 2.1, 2.2, and 2.3, is a variation on an idea of Torunczyk [**18**] which in turn is based on work of Anderson [**1**] and Bessaga-Pelczynski [**3**]. See also [**19**].

DEFINITIONS AND NOTATION FOR §2. $H(Y)$ is the set of homeomorphisms of $Y$. If $U$ is open in $Y$, $\varepsilon: U \to R^+$ is a map and $f \in H(Y)$, let

$$V_U(f, \varepsilon) = \{h \in H(Y) \mid d(f(y), h(y)) + d(f^{-1}(y), h^{-1}(y)) < \varepsilon(y) \text{ for each } y \in Y, \text{ and } h = f \text{ on } Y - U\}.$$

A subset $X$ of $Y$ is *thin in* $Y$ if, for each open $U$ containing $X$ and each map $\varepsilon: U \to R^+$, there exists $h \in V_U(1, \varepsilon)$ such that $h(X) \cap X = \emptyset$.

Let $\mathscr{S}$ be some collection of subsets of $Y$, and let $\mathscr{S}_+$ [resp. $\mathscr{S}_{++}$] be the collection of all finite [resp. countable] unions of closed subsets of elements of $\mathscr{S}$. A subset $B$ of $Y$ is a *pseudo-boundary for* $\mathscr{S}$ *in* $Y$ if $B \in \mathscr{S}_{++}$ and the following *absorption property* holds: For each $S \in \mathscr{S}$, $U$ open in $Y$, and $\varepsilon: U \to R^+$ a map, there exists $h \in V_U(1, \varepsilon)$ such that $h(S \cap U) \subset B$. $P = Y - B$ is a *pseudo-interior for* $\mathscr{S}$ *in* $Y$.

*Possible Axioms for* $\mathscr{S}$:

I (*closed*). The elements of $\mathcal{S}$ are closed in $Y$.
II (*invariant*). The elements of $\mathcal{S}$ are invariant under $H(Y)$.
III (*thin*). The elements of $\mathcal{S}_+$ are thin in $Y$.

THEOREM 2.1 (WEST [19, THEOREM 1]). *Let $\mathcal{S}$ satisfy* I *and* II, *let $B$ and $B'$ be pseudo-boundaries for $\mathcal{S}$ in $Y$, let $U$ be open in $Y$ and let $\varepsilon : U \to R^+$ be a map. Then there exists $f \in V_U(1, \varepsilon)$ such that $f(B \cap U) = B' \cap U$.*

If $B$ is a pseudo-boundary and $T \in \mathcal{S}_{++}$, then $B \cup T$ is also a pseudo-boundary. This with Theorem 2.1 leads to

COROLLARY 2.2. *Let $\mathcal{S}$ satisfy* I *and* II, *let $P$ be a pseudo-interior for $\mathcal{S}$ in $Y$ and let $T \in \mathcal{S}_{++}$. Then $P \cap T$ is strongly negligible in $P$.*

By further considering the implications of Theorem 2.1, one proves

COROLLARY 2.3. *Let $\mathcal{S}$ satisfy* I, II *and* III, *and let $B$ be a pseudo-boundary for $\mathcal{S}$ in $Y$. If $T \in \mathcal{S}_+$, then $B \cap T$ is strongly negligible in $B$. Moreover, if $T'$ is any closed subset of $Y$ which lies in $B|$ then $T'$ is strongly negligible in $B$.*

COROLLARY 2.3 applies in particular to compact subsets of $B$.

3. **The universal pseudo-boundaries in $E^n$.** A closed subset $X$ of a space $Y$ is a $Z_m$-set ($m$ an integer $\geqq 0$) if, for every nonempty $m$-connected open set $U$ in $Y$, $U - X$ is nonempty and $m$-connected. A closed subset $X$ of $E^n$ is a *strong $Z_m$-set* ($-1 \leqq m < n$) if, for each compact subpolyhydron $P$ of $E^n$ having dimension $\leqq m + 1$, and each $\varepsilon > 0$, there is an $\varepsilon$-push (see [7]) $h$ of $(E^n, X \cap P)$ such that $h(X) \cap P = \varnothing$. Let $\mathcal{M}_n^k$ be the family of all strong $Z_{n-k-2}$-sets in $E^n$, $-1 \leqq k < n$.

EXAMPLES OF STRONG $Z_{n-k-2}$-SETS IN $E^n$. Any $k$-dimensional subpolyhedron of $E^n$; any compact subset whose complement is 1-ULC, provided $k \leqq n - 3$ and $n \neq 4$; if $k = n - 1$ or if $1 \neq k = n - 2$, then strong $Z_{n-k-2}$ is equivalent to dimension $\leqq k$; strong $Z_{n-k-2}$ always implies dim $\leqq k$.

LEMMA 3.1. *$\mathcal{M}_n^k$ satisfies* I *and* II. *If $n \geqq 2k + 1$, $\mathcal{M}_n^k$ also satisfies* III.

Obviously I holds. The hardest part of showing II is the case $k \leqq n - 3$ and $n \geqq 5$: For this, one uses engulfing theorems in [7], [**8**] or alternatively [**16**], to show that strong $Z_{n-k-2}$ is equivalent to $Z_1$ and dimension $\leqq k$; the latter is indeed an invariant property. The remaining special cases are derived from [4], [5], [10] and [12]. A general position argument verifies III.

*A pseudo-boundary.* Partition $E^1$ into intervals whose endpoints are $l/3^{i-1}$, $i \geqq 1$, $l$ an integer. Regarding $E^n$ as a product, let $J_i$ be the product cell complex partitioning $E^n$, its $k$-skeleton being $J_i^k$. Let $K_i$ be the subcomplex of $J_{i+1}$ generated by the cells which touch cells of $J_i^k$. Let $B_i(\mathscr{M}_n^k) = \bigcup_{j=i}^{\infty} |K_j|$ and let $B(\mathscr{M}_n^k) = \bigcup_{i=1}^{\infty} B_i(\mathscr{M}_n^k)$. The intersection of $B_i(\mathscr{M}_n^k)$ with an $n$-cell of $J_i$ is often called a Menger universal $k$-dimensional compactum in $E^n$ (its universality, which we do not need, is proved in [**17**]). We therefore call $B(\mathscr{M}_n^k)$ the *universal k-dimensional pseudo-boundary in* $E^n$.

THEOREM 3.2. *$B(\mathscr{M}_n^k)$ is a pseudo-boundary for $\mathscr{M}_n^k$ in $E^n$.*

Clearly $B(\mathscr{M}_n^k) \in (\mathscr{M}_n^k)_{++}$. By a method of Bothe [**6**], each element of $\mathscr{M}_n^k$ can be (ambiently) pushed into $B_i(\mathscr{M}_n^k)$ for any $i$. Bothe's methods can be adapted to show that $B(\mathscr{M}_n^k)$ has the absorption property as required.

Besides Theorem 1.1 (in which one takes $B^k = B(\mathscr{M}_n^k)$ and $P^{n-k-1} = E^n - B(\mathscr{M}_n^k) = P(\mathscr{M}_n^k)$) we have other interesting negligibility theorems which reflect the situation in $I^\omega$ (compare Theorems 0 and 1 of [**2**]).

THEOREM 3.3. *Assume $(n, k) \neq (3, 1)$, $(4, 0)$ or $(4, 1)$. A closed subset of $P(\mathscr{M}_n^k)$ is strongly negligible in $P(\mathscr{M}_n^k)$ if and only if it is a $Z_{n-k-2}$-set in $P(\mathscr{M}_n^k)$. An arbitrary subset is strongly negligible if and only if it is the countable union of $Z_{n-k-2}$-sets.*

THEOREM 3.4. *A closed subset of $B(\mathscr{M}_{2k+1}^k)$, $k \neq 1$, is strongly negligible in $B(\mathscr{M}_{2k+1}^k)$ if and only if it is a $Z_{k-1}$-set in $B(\mathscr{M}_{2k+1}^k)$.*

Note that Theorems 3.3 and 3.4 are "intrinsic" theorems: $E^n$ is nowhere mentioned.

4. **The polyhedral pseudo-boundaries in $E^n$, $n \neq 4$.** Let $\mathscr{P}_n^k$ be the family of all tame $k$-dimensional polyhedra in $E^n$. From §3, we deduce

LEMMA 4.1. *$\mathscr{P}_n^k$ satisfies* I *and* II. *If $n \geqq 2k + 1$, $\mathscr{P}_n^k$ also satisfies* III.

The polyhedral $k$-dimensional pseudo-boundary in $E^n$ is the set $\tilde{B}_n^k$ defined in §1. Here we will call it $B(\mathscr{P}_n^k)$. We have

THEOREM 4.2. *If $n \neq 4$, $B(\mathscr{P}_n^k)$ is a pseudo-boundary for $\mathscr{P}_n^k$ in $E^n$.*

To prove Theorem 4.2 one needs the "Hauptvermutung" for $E^n$ ([**4**], [**11**] and [**14**]) which is unknown when $n = 4$.

The negligibility Theorem 1.2 follows as in §2, though 1.2(4) seems to require a codimension 3 taming theorem (see Theorem 4 of [**9**] and Theorem 1 of [**15**]).

5. **The infinite-dimensional analogy.** The usual pseudo-boundary of $I^\omega$ is itself the countable union of copies of $I^\omega$; in other words, it is the coun-

ble union of universal compacta in $I^\omega$. Compare with the universal $k$-dimensional pseudo-boundary $B(\mathscr{M}_n^k)$ of §3 which is the countable union of universal $k$-dimensional compacta in $E^n$. There is also a smaller pseudo-boundary in $I^\omega$, defined by Anderson in [**2**]. This one is the countable union of finite-dimensional cubes. Compare with the polyhedral $k$-dimensional pseudo-boundary $B(\mathscr{P}_n^k)$ of §4 which is the countable union of $k$-dimensional cubes. So far a good analogy.

But more is known about the infinite-dimensional case. While the two pseudo-boundaries in $I^\omega$ are not equivalent (in one case take $\mathscr{S}$ to be the family of all $Z$-sets in $I^\omega$, in the other take $\mathscr{S}$ to be the family of all tame polyhedra in $I^\omega$), the corresponding pseudo-interiors are both homeomorphic to Hilbert space $l_2$ (see [**2**]). Letting $P(\mathscr{M}_n^k) = E^n - B(\mathscr{M}_n^k)$ and $P(\mathscr{P}_n^k) = E^n - B(\mathscr{P}_n^k)$, the analogy suggests

CONJECTURE. *If $n \leqq 2k + 1$, then $P(\mathscr{M}_n^k)$ and $P(\mathscr{P}_n^k)$ are homeomorphic.*

The conjecture is easily proved when $k = 0$ and $n = 1$. R. D. Anderson has pointed out that it is false if $k = 0$ and $n \geqq 2$. The restriction $n \leqq 2k + 1$ seems reasonable.

6. **Pseudo-boundaries in topological manifolds.** Let $M$ be a separable metrizable $n$-manifold without boundary. A *euclidean chart* for $M$ is a pair $(h, W)$ where $W$ is open in $M$ and $h: E^n \to W$ is a homeomorphism. A closed subset $X$ of $M$ is a *local strong $Z_{n-k-2}$-set in $M$* (resp. *local tame $k$-dimensional polyhedral set in $M$*) if for each $x \in X$ there is a euclidean chart $(h, W)$ with $x \in W$ and $h^{-1}(X)$ a strong $Z_{n-k-2}$-set in $E^n$ (resp. a tame $k$-dimensional polyhedron in $E^n$). Let $\mathscr{M}_M^k$ (resp. $\mathscr{P}_M^k$) be the family of all local strong $Z_{n-k-2}$-sets in $M$ (resp. all local tame $k$-dimensional polyhedral sets in $M$).

LEMMA 6.1. *$\mathscr{M}_M^k$ and $\mathscr{P}_M^k$ satisfy* I *and* II; *if $n \geqq 2k + 1$ and $(n, k) \neq (4, 0)$ or $(4, 1)$, they also satisfy* III.

Let $\{(h_i, W_i)\}$ be a countable set of euclidean charts such that $M = \bigcup_{i=1}^\infty W_i$. Define $B(\mathscr{M}_M^k) = \bigcup_{i=1}^\infty h_i(B(\mathscr{M}_n^k))$ and $B(\mathscr{P}_M^k) = \bigcup_{i=1}^\infty (B(\mathscr{P}_n^k))$.

THEOREM 6.2. *If $(n, k) \neq (4, 0)$ or $(4, 1)$, then $B(\mathscr{M}_M^k)$ is a pseudo-boundary for $\mathscr{M}_M^k$ in $M$. If $n \neq 4$, $B(\mathscr{P}_M^k)$ is a pseudo-boundary for $\mathscr{P}_M^k$ in $M$.*

2.1, 6.1 and 6.2 imply that $B(\mathscr{M}_M^k)$ and $B(\mathscr{P}_M^k)$ are independent of the charts $\{(h_i, W_i)\}$ up to homeomorphism of $M$.

Thus the notions of universal and polyhedral pseudo-boundaries can be sensibly extended to manifolds. Negligibility theorems follow as in previous sections.

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