BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

STATISTICAL MECHANICS ON A COMPACT SET WITH Z^ν ACTION SATISFYING EXPANSIVENESS AND SPECIFICATION

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Communicated by Jacob Feldman, April 27, 1972

1. **Introduction.** When Sinai [11], [12] and Bowen [1], [2] studied invariant measures for an Anosov diffeomorphism, or on basic sets for an Axiom A diffeomorphism, they encountered problems reminiscent of statistical mechanics (see [10, Chapter 7]). Sinai [13] has in fact explicitly used the techniques of statistical mechanics to show that an Anosov diffeomorphism does not in general have a smooth invariant measure.

We rewrite here a part of the general theory of statistical mechanics for the case of a compact set Ω satisfying expansiveness and the specification property of Bowen [1]. Instead of a Z action we consider a Z^ν action as is usual in lattice statistical mechanics (where $\Omega = F^{Z^\nu}$ with F a finite set). This rewriting presents a number of technical problems, but the basic ideas are contained in the papers of Gallavotti, Lanford, Miracle-Sole, Robinson, and Ruelle [5], [7], [8], [9], etc.

2. **Notation and assumptions.** Given integers $a_1, \ldots, a_\nu > 0$, let $Z^\nu(a)$ be the subgroup of Z^ν with generators $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, a_\nu)$. We write also

$$\Lambda(a) = \{m \in Z^\nu : 0 \leqq m_i < a_i\},$$

$$\Pi(a) = \{x \in \Omega : Z^\nu(a)x = \{x\}\}.$$

If (Λ_α) is a directed family of finite subsets of Z^ν, $\Lambda_\alpha \nearrow \infty$ means $\operatorname{card} \Lambda_\alpha \to \infty$ and $\operatorname{card}(\Lambda_\alpha + F)/\operatorname{card} \Lambda_\alpha \to 1$ for every finite $F \subset Z^\nu$. In particular $\Lambda(a) \nearrow \infty$ when $a \to \infty$ (i.e. when $a_1, \ldots, a_\nu \to \infty$).

Let Z^ν act by homeomorphisms on the metrizable compact set Ω, and let d be a metric on Ω. $C(\Omega)$ is the Banach space of real continuous functions on Ω with the sup norm, and $C(\Omega)^*$ the space of real measures on Ω with the vague topology. The two assumptions below will be made throughout what follows.

2.1. *Expansiveness. There exists* $\delta^* > 0$ *such that*

$$(d(mx, my) \leqq \delta^* \text{ for all } m \in Z^\nu) \Rightarrow (x = y).$$

2.2. *Specification. Given* $\delta > 0$ *there exists* $p(\delta) > 0$ *with the following property. If* (Λ_l) *is a family of subsets of* $\Lambda(a)$ *such that the sets* $\Lambda_l + Z^\nu(a)$

AMS 1970 *subject classifications.* Primary 58F99, 28A65, 54H20; Secondary 82A05.

have mutual (Euclidean) distances $p(\delta)$, *and if* (x_l) *is a family of points of* Ω, *there exists* $x \in \Pi(a)$ *such that*

$$d(mx_l, mx) < \delta$$

for all $m \in \Lambda_l$, *all* l.

If Ω is a basic set for an Axiom A diffeomorphism ($\nu = 1$), it is known that expansiveness [**14**] holds, and that specification [**1**] holds for some iterate of the diffeomorphism.

3. **Pressure and entropy.** Letting $\delta > 0$, we say that $E \subset \Omega$ is (δ, Λ)-separated if

$$((x, y) \in E \text{ and } d(mx, my) < \delta \text{ for all } m \in \Lambda) \Rightarrow (x = y).$$

Let $\phi \in C(\Omega)$. Given $\delta > 0$ and a finite $\Lambda \subset Z^\nu$, or given $a = (a_1, \ldots, a_\nu)$ we introduce the "*partition functions*"

$$Z(\phi, \delta, \Lambda) = \max_E \sum_{x \in E} \exp \sum_{m \in \Lambda} \phi(mx),$$

where the max is taken over all (δ, Λ) separated sets, or

$$Z(\phi, a) = \sum_{x \in \Pi(a)} \exp \sum_{m \in \Lambda(a)} \phi(mx).$$

3.1. THEOREM. *If* $0 < \delta < \delta^*$, *the following limits exist*:

$$\lim_{\Lambda \to \infty} \frac{1}{\text{card } \Lambda} \log Z(\phi, \delta, \Lambda) = P(\phi),$$

$$\lim_{a \to \infty} \frac{1}{\text{card } \Lambda(a)} \log Z(\phi, a) = P(\phi),$$

where P *defines a real convex function on* $C(\Omega)$ *such that*

$$|P(\phi) - P(\psi)| \leqq \|\phi - \psi\|;$$

P is called the pressure.

Other definitions of P, using open coverings or Borel partitions of Ω, are possible.

Let $\mathscr{A} = (A_j)_{j \in J}$ be a finite Borel partition of Ω, and Λ a finite subset of Z^ν. We denote by $\mathscr{A}^\Lambda$ the partition of Ω consisting of the sets $A(k) = \bigcap_{m \in \Lambda} (-m) A_{k(m)}$ indexed by maps $k : \Lambda \to J$. We write

$$S(\mu, \mathscr{A}) = -\sum_j \mu(A_j) \log \mu(A_j).$$

Let I be the (convex compact) set of Z^ν invariant probability measures on Ω.

3.2. THEOREM. *If* $\mathscr{A}$ *consists of sets with diameter* $\leqq \delta^*$ *and* $\mu \in I$, *then*

$$\lim_{\Lambda\to\infty} \frac{1}{\text{card}\,\Lambda} S(\mu, \mathscr{A}^\Lambda) = \inf_\Lambda \frac{1}{\text{card}\,\Lambda} S(\mu, \mathscr{A}^\Lambda) = s(\mu).$$

This limit is finite $\geqq 0$, *and independent of* $\mathscr{A}$. *Furthermore, s is affine upper semi-continuous on* I; *s is called the entropy.*

For $v = 1$, this is the usual definition of the measure theoretic entropy. Specification is not used in the proof of Theorem 3.2.

4. **Variational principle and equilibrium states.** Let I be the set of $\mu \in C(\Omega)^*$ such that

$$P(\phi + \psi) \geqq P(\phi) + \mu(\psi) \qquad \text{for all } \psi \in C(\Omega).$$

Those μ are called *equilibrium states* for ϕ.

4.1. THEOREM. *The following variational principle holds:*

$$(*) \qquad P(\phi) = \max_{\mu\in I} [s(\mu) + \mu(\phi)].$$

The maximum is reached precisely for $\mu \in I_\phi$ *(in particular* $I_\phi \subset I$*). The set* I_ϕ *is not empty; it is a Choquet simplex, and a face of* I [3]. *There is a residual subset* D *of* $C(\Omega)$ *such that* I_ϕ *consists of a single point* μ_ϕ *if* $\phi \in D$. *For all* $\mu \in I$,

$$s(\mu) = \inf_{\phi\in C(\Omega)} [P(\phi) - \mu(\phi)].$$

If Ω is a basic set for an Axiom A diffeomorphism it is known [2] that $0 \in D$, and $(*)$ for $\phi = 0$ is related to the fact that the topological entropy is the sup of the measure theoretic entropy [4], [6]. Further results on D have been obtained for Anosov diffeomorphisms using methods of statistical mechanics [13].

4.2. THEOREM. *Let* $\mu_{\phi,a}$ *be the measure on* Ω *which is carried by* $\Pi(a)$ *and gives* $x \in \Pi(a)$ *the mass*

$$\mu_{\phi,a}(\{x\}) = Z(\phi, a)^{-1} \exp \sum_{m\in\Lambda(a)} \phi(mx).$$

If μ *is a (vague) limit point of the* $(\mu_{\phi,a})$ *when* $a \to \infty$, *then* $\mu \in I_\phi$. *In particular, if* $\phi \in D$,

$$\lim_{a\to\infty} \mu_{\phi,a} = \mu_\phi.$$

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