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THREE CHARACTERISTIC CLASSES MEASURING THE OBSTRUCTION TO PL LOCAL UNKNOTEDNESS

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$(M, \partial M) \subset (N, \partial N)$ denotes a PL embedding of oriented PL manifolds, with $M \cap \partial N = \partial M$, $\operatorname{codim}_N(M) = 2$. Two such embeddings i_0, i_1 are concordant if there exists a PL embedding $(M \times I, \partial M \times I, M \times \partial I) \subset (N \times I, \partial N \times I, N \times \partial I)$ restricting to i_0, i_1 on $M \times 0, M \times 1$, respectively. The embedding $(M, \partial M) \subset (N, \partial N)$ is locally flat if every point $x \in M$ has a PL ball D^n for its neighborhood such that $D^n \cap M \subset D^n$ is PL conjugate to the standard embedding $D^{n-2} \subset D^n$.

This is the problem considered here: When is the embedding $(M, \partial M) \subset (N, \partial N)$ concordant to a locally flat embedding?

Let G_k denote the (geometric) group of concordism classes of locally flat knots having dimension k. Stabilizing, and factoring out by the periodicity isomorphism, we get a Z_4-graded group, which is denoted (somewhat akwardly) as G_*. $K^*_{F/PL}$, $K^*_{\overline{F/\mathrm{TOP}}}$ denote the cohomology theories having F/PL, $\overline{F/\mathrm{TOP}}$ for their zeroth loop spectrum [**11**].

THEOREM. *There are characteristic classes $\theta(M, N) \in H^2(M, G_3)$, $\beta(M, N) \in K^0_{F/PL}(M)$, $\gamma(M, N) \in K^0_{\overline{F/\mathrm{TOP}}}(M, G_{*+1})$ satisfying the following*:

(a) *These classes depend only on the concordism class of the embedding $(M, \partial M) \subset (N, \partial N)$.*

(b) *They vanish if and only if $(M, \partial M) \subset (N, \partial N)$ is concordant to a locally flat embedding.*

Construction of θ, β, γ.

θ. For each simplex $\Delta^k \in M$ there are cells $D_M(\Delta^k)$, $D_N(\Delta^k)$—the dual cells to Δ^k in M, N, respectively. These satisfy $D_N(\Delta^k) \cap M = D_M(\Delta^k)$; $D_m(\Delta^k) \subset D_N(\Delta^k)$ is a codimension 2 embedding of discs. Try to concord $(M, \partial M) \subset (N, \partial N)$ to a locally flat embedding by inductively doing so for the embeddings $(D_M(\Delta^k) \subset D_N(\Delta^k)$. The first possible nonvanishing obstruction appears as a cocycle defined on the 2-dimensional dual cells in M. This represents $\theta(M, N) \in H^2(M, G_3)$ [**10**].

γ. Let (R, R_∂) denote a regular neighborhood for $(M, \partial M) \subset (N, \partial N)$. $\dot{R}$ denotes its topological boundary in N. There is a linear bundle τ defined over M, having D^2 for fiber, and an integral-homology equivalence

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$f:(R,\dot{R}) \to (\tau,\dot{\tau})$. Using f in conjunction with the new surgery groups of Cappell-Shaneson, a splitting obstruction can be defined along any singular Z_q-manifold $g:(V,\delta V) \to M$ as in [6]. By the universal coefficient theorem for $K^*_{\overline{F/\mathrm{TOP}}}$, these represent an element $\gamma(M,N) \in K^0_{\overline{F/TOP}}(M, G_{*+1})$.

β. For large r, $f \times 1_{D^r}$: $(R,\dot{R}) \times (D^r,\partial D^r) \to (\tau,\dot{\tau}) \times (D^r,\partial D^r)$ represents an element $\beta(M,N) \in K^0_{F/PL}(M)$.

REMARK 1. There is a classifying space $BSPL^{\sim}(2)$ for codimension 2 PL thickenings of PL manifolds. There are universal characteristic classes

$$\theta \in H^2(BSPL^{\sim}(2), G_3),$$
$$\beta \in K^0_{F/PL}(BSPL^{\sim}(2)),$$
$$\gamma \in K^0_{\overline{F/\mathrm{TOP}}}(BSPL^{\sim}(2), G_{*+1}).$$

Modulo low-dimensional complications (over the 4 skeleton of $BSPL^{\sim}(2)$) there is a homotopy equivalence

$$BSPL^{\sim}(2) \xrightarrow{f \times \gamma} BSO(2) \times (\overline{F/\mathrm{TOP}} \otimes G_{*+1}).$$

REMARK 2. The classes θ, β, γ are related as follows. The geometric stabilization of θ is determined by the restriction of γ to the 2-skeleton of M. The stabilization of β is a direct summand of γ.

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