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DIFFERENTIABLE ACTIONS ON 2n-SPHERES

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Introduction. It is shown in [4] that there is an infinite family of semifree Z_m actions on odd dimensional homotopy spheres. There is also an infinite family of semifree S^1 actions on odd dimensional homotopy spheres (see [2], [5]). On the other hand, it is announced in [2] that there are only finitely many inequivalent semifree S^1 actions on even dimensional homotopy spheres. Hence it is interesting to know whether the same phenomenon occurs for Z_m actions. The study of the Atiyah-Singer G-signature theorem and an exact sequence of M. Rothenberg leads to the discovery of an infinite family of semifree Z_m actions on even dimensional homotopy spheres which, to the best of author's knowledge, is not previously known. The main result is the following:

THEOREM. *There exist infinitely many inequivalent semifree Z_m actions on S^{2n} with fixed point set S^{2r} for $r \leqq n/3m \neq 2$.*

The author would like to thank Professor M. Rothenberg; most ideas of this paper are due to him.

Sketch of the proofs. For $m \neq 2$, let $\rho : Z_m \to U(n-r)$ be a unitary fixed point free representation of complex dimension $n - r$ without eigenvalue -1. Then $\rho = \sum_{n_j > 0} n_j t^{a_j}$ where $1 \leqq a_1 < \ldots < a_s \leqq m - 1$, $a_j \neq m/2$, and t is the basic complex one dimensional representation of Z_m defined to be multiplication by $\exp(2\pi i/m)$. Let $C(\rho)$ be the centralizer of $\rho(Z_m)$ in $U(n-r)$. Define a map $A : \pi_{2r-1}(C(\rho)) \to C^{Z_m - 1}$ as follows.

For $f: S^{2r-1} \to C(\rho)$, let η be the vector bundle over S^{2r} with f as characteristic map. Let Z_m act on η via ρ. It is clear that Z_m acts freely on $S(\eta)$. By a theorem in [3], for some k, there exists a manifold W supporting a free Z_m action and $\partial W = kS(^K)$. Define $A(f)(g) = {}^{k-1}\,\mathrm{Sign}(g, W \cup kD(\eta))$ for $g \in Z_m$ where $\mathrm{Sign}(g, M)$ is the character of the G-signature of a G-manifold M (see [1]). It is easy to see that A is well defined.

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Let

$$\prod \frac{\tanh i\theta/2}{\tanh(x_j + i\theta)/2} = \sum \Phi_{i_1 \cdots i_s}(\theta) c_{i_1} \cdots c_{i_s}$$

where c_i is the ith Chern class. Then an elementary calculation shows that

$$\Phi_{i_1 \cdots i_s}(-\theta) = (-1)^{i_1 + \cdots + i_s} \Phi_{i_1 \cdots i_s}(\theta).$$

$C(\rho)$ is isomorphic to $U(n_1) \times \ldots \times U(n_s)$. Let $f = (f_1, \ldots, f_s)$. Then by the G-signature theorem [**1**],

$$A(f)(v^k) = K_k \cdot \left(\sum_{j=1}^{s} \Phi_r\left(2ka_j\pi/m \right) f_j * c_r \right) [S^{2r}]$$

where $Z_m = \langle v \rangle$ and the K_k are constants depending only on k. Since a stable bundle over a sphere is determined by its Chern classes,

$$\text{rank Im } A = \text{rank}(\Phi_r(2ka_j\pi/m))_{k=1,\cdots,m-1,\, j\in\Lambda} \quad \text{where } \Lambda = \{j | n_j \geqq r\}.$$

Since $r \leqq n/3$ we may choose $n_1 \geqq r$ and $n_2 \geqq r$ and $n_1 + n_2 + r = n$. Take $\rho = n_1 t + n_2 t^{m-1}$. Then

$$\pi_{2r-1}(U(n_1) \times U(n_2)) \otimes C = C \oplus C$$

and

$$A(f)(v^k) = K_k \cdot \Phi_r(2k\pi/m)(f_1 * c_r + (-1)^r f_2 * c_r)[S^{2r}].$$

Hence rank Im $A \leqq 1$ or equivalently rank ker $A \geqq 1$.

Now we consider the following exact sequence where notations are the same as in [**5**]:

$$0 \to CS^{2n}(\rho) \otimes C \to \pi_{2r-1}(C(\rho)) \otimes C \xrightarrow{\psi} RS^{2n-1}(\rho) \otimes C \to \ldots$$

where $RS^{2n-1}(\rho) \otimes C$ can be identified as $\tilde{L}_{2n}(Z_m) \otimes C \oplus \pi_{2r-1}(O(2n-2r)) \otimes C$, $\tilde{L}_{2n}(Z_m)$ is the reduced Wall group, and $\tilde{L}_{2n}(Z_m) \otimes C$ can be identified as a subgroup of $C^{Z_m - 1}$ (see [**6**, p. 168]). Now up to isomorphism, ψ is the same map as $A \oplus i_*$ where $i_*: \pi_{2r-1}(C(\rho)) \to \pi_{2r-1}(O(2n-2r))$ is induced by the inclusion. Hence for $\rho = n_1 t + n_2 t^{m-1}$, rank ker $A \geqq 1$ and ker $i_* \supset$ ker A. Therefore rank $CS^{2n}(\rho) \otimes C \geqq 1$. If $m = 2$ then $A = 0$ and rank $CS^{2n}(\rho) = 1$. Since $\bar{S}^{2n}(\rho) \to CS^{2n}(\rho)$ is an epimorphism [**4**] where $\bar{S}^{2n}$ is the group of semifree Z_m actions on homotopy $2n$-spheres with local representation ρ, we have proved our theorem.

REMARK. If $\Phi_r(2k\pi/m) \neq 0$ for some k, then there are only finitely many inequivalent semifree Z_m actions on homotopy $2n$-spheres with fixed point set homotopy $2r$-spheres for $r > n/3$. This is the case for $r = 3$ and $m \geqq 3$. Hence the restriction $r \leqq n/3$ is best possible.

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