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HOW A MINIMAL SURFACE LEAVES AN OBSTACLE[1]

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ABSTRACT. We announce that the function of least area among all functions defined in a convex domain, vanishing on its boundary, and constrained to lie above a concave analytic obstacle leaves the obstacle along an analytic curve.

We announce a result about the curve of separation determined by the solution to a variational inequality. A strictly convex domain Ω with smooth boundary $\partial\Omega$ is given in the $z = x_1 + ix_2$ plane together with a smooth function $\psi(z)$ which assumes a positive maximum in Ω and is negative on $\partial\Omega$. Let K denote the closed convex set of Lipschitz functions v satisfying $v \geqq \psi$ in Ω and $v = 0$ on $\partial\Omega$. Let us denote by u the function of K which minimizes area among all functions of K; that is

$$(1) \qquad u \in K: \quad \int_\Omega \frac{u_{x_j}}{(1 + |u_x|^2)^{1/2}} (v - u)_{x_j} \, dx \geqq 0, \qquad v \in K.$$

The existence of such u, actually satisfying $u \in H^{2,q}(\Omega) \cap C^{1,\lambda}(\bar{\Omega})$, $1 \leqq q < \infty$, $0 < \lambda < 1$, was shown in the work of H. Lewy and G. Stampacchia [7] and also in M. Giaquinta and L. Pepe [1]. For u there is a set of coincidence I consisting of the points $z \in \Omega$ where $u(z) = \psi(z)$. Let us call

$$(2) \qquad \Gamma(u) = \Gamma = \{(x_1, x_2, x_3): x_3 = u(z) = \psi(z), z \in \partial I\}$$

the "curve" of separation.

Up to this time it has only been known that when ψ is smooth and strictly concave, Γ is a Jordan curve [2]. On the other hand, the corresponding problem for the $u \in K$ minimizing the Dirichlet integral has been thoroughly studied by H. Lewy and G. Stampacchia [6]. We wish to announce here the

THEOREM. *Let ψ be analytic and strictly concave. Let u be the solution of* (1). *Then $\Gamma(u)$ is an analytic Jordan curve* (*as a function of its arc length parameter*).

The demonstration relies on the resolution of a system of differential equations and the utilization of the system to extend analytically a con-

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formal representation of the minimal surface which is the graph of u in the subset of Ω where $u > \psi$. This is the idea of Hans Lewy (cf. for example [4], [5]). To derive the system of equations requires knowing that u has bounded second derivatives, which was shown in [3]. In order to identify the solution, we prove first that Γ is rectifiable.

In order to present a precise statement of this first step of the smoothness of Γ, let us introduce some notations. Set $\Sigma = \{x \in R^3 : x_3 = u(z), z \in \Omega\}$, $S = \{x \in R^3 : x_3 = u(z), z \in \Omega - I\} \subset \Sigma$, and $D = \{|\zeta| < 1\}$. Let $X : D \to \Sigma$ be a uniformization (conformal representation) of the $C^{1,\lambda}$ surface Σ with $X(0) = P \in \Gamma$, a fixed point of Γ.

THEOREM. *Let $f = f_\varepsilon$ be a conformal mapping of $G = \{\operatorname{Im} t > 0, |t| < 1\}$ onto a Jordan domain $f_\varepsilon(G)$ containing $\{\zeta : X(\zeta) \in S, |\zeta| < \varepsilon\}$ such that $f_\varepsilon : (-1, 1) \to X^{-1}(\Gamma)$ and $f_\varepsilon(0) = 0$. Then there exists an $\varepsilon > 0$ such that $f_\varepsilon \in C^1(\bar{G})$.*

From this it is clear that the conformal representation $X\{f(t)\}$ provides, locally, a C^1 representation of Γ. The proof of the theorem relies on the strict concavity of ψ and results of [2] to show that $f' \in L^q(G)$ for a $q > 2$.

To prove the second theorem mentioned, we assume only that $\psi \in C^3(\Omega)$ and is strictly concave. Hence we present a method to rectify curves determined by variational inequalities.

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