

Not only are the frontiers of dimension theory expanding, but its foundations and principles are becoming simpler and more elegant. P. Ostrand [5] has recently developed a novel approach to the study of Lebesgue covering dimension which allows one to prove in a simple and elegant fashion many of the classical theorems in dimension theory, including the theorem that $\dim X = \text{Ind } X$ for metric spaces. It seems likely that his approach will ultimately lead to a greatly simplified development for the theory of Lebesgue covering dimension. Despite the elegance and efficiency of Nagami's development I would be disappointed if future works on dimension theory do not make use of Ostrand's ideas to produce an even more exciting and transparent approach to dimension theory.

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Diffusion Processes and their Sample Paths by Kiyosi Ito and Henry P. McKean, Jr., Springer-Verlag, Berlin, 1965.

This book is an excellent illustration of the thesis that once the foundation for solving a basic problem is laid, no matter how complicated the solution may be the problem will find authors equal to the task. In the present case, the historical genesis of the problem goes back to the work of W. Feller in the early 1950's, on characterizing the most general diffusion operator in one dimension. In this work the role of probability was largely confined to motivation and the aim was to characterize certain differential operators by purely analytic axioms. At the time when Feller's treatment of the subject reached its most final form (Illinois J. Math. 1957, 1958) one finds the issue stated as that

of finding the explicit form of the most general linear operator A which
 (a) has domain and range in the continuous functions $f(x)$ on an open interval I ,

(b) is of local character: $f \in D_A$ and $f \equiv 0$ near s_0 imply $Af(s_0) = 0$,

(c) has the strong minimum property: if $f \in D_A$ and f has a local minimum at s_0 then $Af(s_0) \geq 0$,¹ and

(d) is nonsingular: near each $s_0 \in I$ there exist both increasing and decreasing $f \in D_A$ with $Af(s_0) > 0$.

Under these hypotheses, Feller showed that there exists a continuous increasing scale $x(s)$ and a right-continuous function $m(s)$ such that $Af = (d/dm)(d^+/dx^+)f$ for all $f \in D_A$. While this was a reasonably definitive result from an analytic standpoint it was always perfectly clear that it was not sufficient probabilistically to describe all diffusion processes. In particular, obvious possibilities for Markov processes with continuous paths were, for example, to admit a right-translation point s_0 where an infinitesimal generator would have a singularity of the form cd/ds ($c > 0$), violating (d), or to permit the paths to terminate at a constant rate $c > 0$ which would add a term $-c$ to the generator in violation of (c). From this standpoint, also, assumption (a) would appear arbitrary since at a translation point s_0 only points to one side would be relevant to the local generator.

The probabilistic concepts which were needed in formulating the underlying problem had been developed in the work of J. L. Doob, G. A. Hunt, and others on the Markov property. In the classical assumption of independence of past and future given present it was insufficient simply to let the "present" be a fixed time. It had to be any "stopping time," i.e., time determined by its own past history. The resulting "strong Markov property" was already present in all of the reasonable known processes, but its explicit formulation was to have much importance in arriving at the appropriate definition of a diffusion. As subject of the present book, a one-dimensional diffusion (p. 84) is any strict (i.e., strong) Markov process $x(t)$ on $Q \cup \{\infty\}$ where Q is an interval of the closed real line, with sample space consisting of all functions $w = x(\cdot)$ which are continuous for $0 \leq t < m_\infty(w)$ and equal to ∞ thereafter (where $0 < m_\infty$ is the "lifetime"), and having a time-homogeneous probability function P_a with $P_a\{x(0) = a\} = 1$ for all a . The problem which could then be formulated and solved had two parts: (a) to obtain the explicit infinitesimal generators of the semigroups $T_t f(a) = E_a f(x(t))$ of all such processes, acting on appropriate determining spaces of functions; and (b) starting with the elementary analytical invariants present

¹ This was relaxed in Feller's latest paper (1959) by requiring $f(s_0) < 0$.

in such a generator, to reconstruct the measures P_a of the unique corresponding diffusion process.

For part (a) the work of W. Feller served as a guide, but a powerful further tool was found by E. B. Dynkin in the weak generator G and the "Dynkin's formula"

$$E(u(x(m)) - u(x(0))) = E \int_0^m Gu(x(t)) dt;$$

$u \in D_G$, $E_1(m) < \infty$, used in computing G . It is this generator, sometimes introduced in the form $G = \alpha - \mathcal{G}_\alpha^{(-1)}$ where

$$\mathcal{G}_\alpha f(\cdot) = \int_0^\infty e^{-\alpha t} E_\cdot f(x(t)) dt, \quad \alpha > 0,$$

is the resolvent, which is used throughout the book. The space of functions employed here is not restricted by continuity, but consists of all bounded Borel functions f which satisfy for all $a \in Q$ the two identities $\lim_{b \rightarrow a} f(b) = f(a)$ if $P_a\{m(a_\pm) = 0\} = 1$, where $m(a_\pm)$ are the limits as $\varepsilon \rightarrow 0+$ of the passage times (from a) to $a \pm \varepsilon$. The Blumenthal 0-1 Law shows that these probabilities are either 0 or 1.

As for part (b), the earlier work of K. Ito, A. N. Kolmogorov, and most especially of P. Lévy on the local behavior of the Brownian path functions (where $G = \frac{1}{2} d^2/dx^2$) had prepared another powerful method. The idea of Lévy which proved to be the deus ex machina of the constructive method was that of local time. Here the original concern was with the local time at a single point, and Lévy had showed (in his famous way) that for the reflected Brownian motion $x^+(t)$ on $[0, \infty]$ (where the boundary condition $(d^+/dx^+)f(0) = 0$ is imposed on D_G) the derivative

$$l^+(t, x) = \frac{1}{2} \frac{d^+}{dx^+} \int_0^t I_{[0, x]}(x^+(s)) ds$$

exists at $x = 0$ and is continuous in t with probability 1 (where $I_{[0, x]}$ is the usual indicator function). However, this was extended in 1958 by H. Trotter to simultaneous existence and joint continuity of $l(t, x)$ for all (t, x) , where l is defined from the unreflected process $x(t)$.

The idea of using a time substitution $f(t)$ to transform a Brownian motion $x(t)$ into another diffusion $x'(t) = x(f^{(-1)}(t))$ had already occurred to G. A. Hunt in 1957 in a more general situation. Without the use of local time, however, rather laborious methods of approximation had to be used in order to apply it in general (as in [V. A. Volkonskiï, *Random substitution of time in strong Markov processes*, Theor. Probability Appl. 3 (1958), 310–326]). Ito and McKean were probably the first to show that when the generator G has the form $(d/dm)(d^+/dx^+)$ on $(-\infty, \infty)$

then $f(t) = \int l(t, x)m(dx)$ is precisely the time substitution needed to transform $x(t)$ into the required diffusion $x'(t) = x(f^{-1}(t))$, and thereby to give a construction of the latter. However, they proceeded also to treat the most general case. We shall outline the sequence of generalizations employed in this, since it shows the interrelation of generators and time substitutions and also illustrates the method of proceeding from the particular to the general which is characteristic of the whole book.

The first case treated is the nonsingular, conservative one, i.e., for each $x \neq y$ in Q , $P_x\{X(t) = y \text{ for some } t\} > 0$, and $P_x\{X(t) \neq \infty\} = 1$ for all t . After introducing a change of scale, $b(x)$ the generator is $G = (d/dm) \cdot (d^+/db^+)$, and the new space Q is one of 3 types: $(-\infty, \infty)$, $[0, \infty)$, or $[0, 1]$. For the first, both boundaries $\pm\infty$ are inaccessible and we have the situation mentioned before. In the second, 0 requires an additional boundary condition if it is both "exit" and "entrance" (i.e., if $m(0, 1) < \infty$ where $m(db)$ is the measure with distribution $m(b)$) and the possibilities are $Gu(0) = 0$ (absorption) or $m(0)Gu(0) = u^+(0)$ (delayed reflection, where $u^+(0)$ denotes $d^+ u(0)/db^+$ and $m(0)$ is the mass $m\{0\}$). For $Gu(0) = 0$ one sets $m(0) = \infty$ and $0 \cdot \infty = 0$, whence in terms of the reflected Brownian local time $l^+(x, t)$ the formula $f(t) = \int_{0-}^{\infty} l^+(x, t)m(dx)$ defines the necessary time substitution in both cases. The situation on $[0, 1]$ is analogous.

The next case is when the process is nonconservative, but still nonsingular. The "natural scale" $b(x)$ still exists, and the generator interior to Q has the form $Gu(b) = [u^+(db) - u(b)k(db)]/m(db)$, meaning that

$$\int_{b_1^+}^{b_2^+} Gu(b)m(db) = \frac{d^+}{db^+} u \Big|_{b_1}^{b_2} - \int_{b_1^+}^{b_2^+} u(b)k(db),$$

for $b_1 < b_2$, where $k(db)$ is the "killing measure." For the case $Q = [0, \infty)$ and $m(0, 1) < \infty$, one has either the boundary condition $Gu(0) = \alpha u(0)$, $0 \leq \alpha < \infty$, or $m(0)Gu(0) = u^+(0) - u(0)k(0)$, and the construction begins with the conservative process $z(t)$ corresponding to $G = (d/dm)(d^+/db^+)$ and either $Gu(0) = 0$ or $m(0)Gu(0) = u^+(0)$ respectively. To obtain the desired process, the authors introduce the new local times

$$l(t, b) = \frac{d}{dm} \int_0^t I_{[0, b)}(z(s)) ds,$$

and adjoin a "killing time" m_∞ with the conditional distribution,

$$P_a(m_\infty > t | z(\cdot)) = \exp - \left(\int_{0+}^{\infty} l(t, b)k(db) + \alpha((t - m_0) \vee 0) \right)$$

in the first case, where m_0 is the passage time to 0, and

$$P_a(m_\infty > t | z(\cdot)) = \exp - \int_{0^-}^{\infty} l(t, b)k(db)$$

in the second case. The construction is completed by setting $x'(t) = z(t)$ for $t < m_\infty$ and $x'(t) = \infty$ for $t \geq m_\infty$.

The final case is that of a nonconservative and singular diffusion. Roughly speaking the singularities are points at which a process translates in one direction or the other, or else is trapped, so that a preliminary decomposition of Q reduces it to the case when all interior translations are in the same direction, and there are no interior traps. A second decomposition separates out the "regular intervals" (if any) to each of which the previous case applies. The remaining singularity set K_+ is closed (but otherwise arbitrary) and on this set a "shunt scale" $s_+(d\xi)$ and "shunt killing measure" $k_+(d\xi)$ may be assigned so that the added specification for Gu becomes $\int Gu(\xi)s_+(d\xi) = \int u(d\xi) - \int u(\xi)k_+(d\xi)$, the indefinite integrals being restricted to K_+ . Conversely, the conditions which s_+ and k_+ must satisfy in order to qualify as generator components are also provided. The construction of corresponding processes is too involved to describe here, but it involves piecing together the previous cases for the regular intervals, with an added translation on K_+ at rate 1 in the scale s_+ , and a termination in accordance with k_+ . Thus, to quote from p. 164, "a perfect correspondence is obtained between differential operators G on the one hand and nonsingular or singular diffusions on the other."

The portion of the book discussed above is in Chapters 3–5. Most of the remaining 5 chapters are devoted to Brownian motion itself, except for Chapter 6 on local time in the diffusion setting. As any reader will rapidly discover, none of this is easy reading. In order to cover a wide literature in a reasonable expanse the authors have greatly compressed the treatment. The widespread use of dots in formulas, often dots of different sizes and meanings, together with differences conveyed only by slight changes in the type form, makes the text unsparing on the eyes. Much material is placed in "problems" with outlined solutions. The effect is a lack of local redundancy which sometimes leaves the treatment to the mercy of misprints. The reviewer noted 66 of the latter, but this is only an estimate of the actual number.

In spite of these difficulties the book is remarkably successful in conveying a great deal of material, especially if the reader has some prior knowledge thereof. In many cases the main result of a research paper is condensed to a page or two with no loss of rigour. On one-dimensional Brownian motion (Chapters 1,2) the book covers most of Lévy's work,

together with the more recent results of D. Ray on the local time as x varies and of H. P. McKean, Jr., on Brownian excursions and "Bessel processes" (radical components of Brownian motion in $d > 1$ dimensions). While it lacks the air of discovery present in Lévy's comparable *Processus stochastiques et mouvement Brownien*, the present work has a greater degree of finality and virtuosity. Nevertheless, this material is only preparatory to later chapters and is far from exhausting the subject. Kolmogorov's test is not finally established until the end of Chapter 4, where it is combined with a "Dvoretzky-Erdős test" applicable in d dimensions (the method is due to M. Motoo). Turning to Chapter 6 on local times, we find material which was first presented here. The comparison tests of §6.7, enabling one to compute the Hausdorff dimension of the set of zeros of a process from local properties of its speed measure $m(db)$, seem of special interest. Also the use of the local time $l(t, b)$ in formulating the individual ergodic theorem (§6.8) permits treating the most general additive functional $\int l(t, b)\mu(db)$ in place of the usual $\int_0^t f(x(s)) ds$.

The final two chapters on diffusion in several dimensions can be considered either as completing the frame for Chapters 3–5, or as providing an opening to further developments. They show again the main characteristics of the rest of the book, but to an intensified degree. The ingenuity and conciseness seem to increase, but unfortunately the minor gaps and misprints do likewise. As an example, the first page of Chapter 7 asserts a "Dynkin's formula" without any condition to make m finite in the term $E \int_0^m Gu(x(t)) dt$, without which the integral does not exist. However, the only use made of the formula cleverly circumvents this difficulty. On the same page is found the key assumption that the space of continuous functions vanishing at ∞ is stable under the resolvent. With this, much of the analytical apparatus of the one-dimensional case is carried over quite directly to higher dimensions.

Chapter 7 chiefly treats the Brownian motion and its associated Newtonian potential for any domain having a Green function (0 boundary values). A remarkable sequence of thumbnail proofs of important results is given: Newtonian capacities, Gauss's quadratic form, Wiener's test, the Dirichlet problem, Kelvin's principle, etc. N. Ikeda's solution of the Neumann problem in terms of a reflecting Brownian motion in the unit disk requires only a page for the proof. Various applications are made of a useful skew product representation of d -dimensional Brownian motion in terms of a (radial) Bessel process $r(t)$ and an independent spherical Brownian motion in $d - 1$ dimensions with the time substitution $l(t) = \int_0^t r(s)^{-2} ds$. The substantive ending of the chapter comes with the proof that any diffusion with Brownian hitting probabilities has a generator $(Gu)(a) = -e^u(da)/m(da)$ where $e^u(da)$ is the Riesz measure of u

(if u is a potential then $u = \int \mathcal{G} e^u(da)$ where \mathcal{G} is the Newtonian Green function) and $m(da)$ is a fixed speed measure, in complete analogy with the case $d = 1$. This chapter is a “tour de force” with few references to the earlier ones.

The final chapter, 8, returns to the study of a general diffusion, but unlike the other chapters it is rather a statement of problems and principles than an empirically organized collection of facts. The language is that of road maps and speeds, due to W. Feller, but several of the major problems remain unsolved at the present time (it even may be doubted if all of them admit of complete solutions). On the other hand, the continuity of path assumption is used effectively to condense some of the results of probabilistic potential theory, and especially for the proof that the natural capacities of G. A. Hunt are the same for the forward and reversed (adjoint) processes. The chapter terminates with a short introduction to the general boundary theory of diffusions, which has undergone much development since publication,

Sufficient time may by now have elapsed to make it appropriate to give an overview of this extraordinary work. Suppose that one would take a trip on an ideal diffusing particle—where would one go and what would one see? The trip is rough and often confusing. In many cases one cannot see the route because of the superabundance of detail in the landscape. Furthermore, the trip has no clear beginning or final conclusion. Nonetheless, it does describe a significant branch of mathematics with an elegance of taste and a finality unlike any other work on the subject. Whether or not one wishes to make the journey, it is available and the subject is strengthened by its presence. One might, however, express a hope that in a new edition the authors would provide a few more landmarks.

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Celestial Mechanics. I, II by Shlomo Sternberg. Vol. I, 158 pp., Vol. II, 304 pp. W. A. Benjamin, Inc., New York, 1969. \$7.95.

In recent years there has been considerable growth of interest among mathematicians in classical mechanics. The article of S. Smale, *Topology and mechanics*, and S. Mac Lane’s survey in the *Monthly* are indicative of this.

Sternberg’s book has one important virtue: The author deals not only with generalities (such as symplectic structures and so on) but also treats details of difficult concrete problems.

The theory of Hamiltonian perturbations of quasiperiodic motions in