BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

BOOK REVIEWS

Dimension Theory by Keiô Nagami, Volume 37 in the series Pure and Applied Mathematics, Academic Press, New York and London, 1970, 244+xi pp.

This book has a limited scope and is designed to introduce the reader quickly and efficiently to some of the frontiers of research in modern dimension theory. Examples are emphasized which demonstrate the inequalities between the various dimension functions. With only a basic knowledge of general topology a student should be able to read the text without great difficulty except for the appendix on cohomological dimension theory written by Y. Kodama for which a knowledge of Čech cohomology is advisable.

The author has taken great pains in his presentation. The proofs are generally flawless and efficient and the development is purposeful and clear. There is a generous sprinkling of examples which intersperse the text to act as landmarks by which the reader can keep his bearings as the author leads him through newly charted territories. Authors of modern textbooks in higher mathematics would do well to imitate Nagami's careful and concise presentation. He has made his book an adventure as the reader peers with the author as the limits of man's knowledge in this area unfold before him.

The chapter titles indicate the range of topics covered in the book: 1. Theory of Open Coverings; 2. Dimension of Normal Spaces; 3. Dimension of Metric Spaces; 4. Gaps between Dimension Functions; 5. Dimension-Changing Closed Mappings; 6. Product Theorem and Expansion Theorem; 7. Metric-Dependent Dimension Functions; and an appendix, Cohomological Dimension Theory by Yukihiro Kodama. This would certainly not be a complete list of topics in a comprehensive work on dimension theory. For that matter, it is not a complete list of current research topics in this area of mathematics. There has been a restriction of the topics treated according to the taste and interest of the author. Although this restriction is evidently by design, the reader should be aware of its severity. One must search and strain to find an indication that the dimension of the n-cube is n. At last! In a footnote on the bottom of a page, one is referred to Hurewicz and Wallman's *Dimension Theory* for an elegant proof of this result. One meets the classical theorems in dimension theory for separable metric spaces only in the Preface and then they are only named in passing as part of the "classical dimension theory for separable metric spaces" embodied in Hurewicz and Wallman's book. With this book as sole text the reader would be brought to certain frontiers of knowledge, but his perspective would be limited in the process. For this reason one should not be misled into thinking that because this book is the most recent work in dimension theory, that it is therefore all-inclusive and that it supersedes all previous works on the subject. It was clearly not written with this purpose in mind. With this understanding, the book serves a very useful purpose in giving the student and researcher easy access to many important examples and theorems in modern dimension theory, particularly in the theory for nonmetric spaces.

One excellent feature should be mentioned. There are a number of research problems posed at strategic points in the book. This is extremely worthwhile for those who would like to gain some insight into the sorts of things that consume the waking hours of dimension theorists and make their sleep fitful. However, it would have been worthwhile if the history of these problems could have been described briefly and references given. The casual reader might suppose that the problems were simply stated out of the author's imaginative curiosity. Many of the problems stated in the book are "classical" and have a long history in the literature of statement, restatement, and partial solution. Other problems have been more recently posed and, not having stood the test of time, their significance is more doubtful. A little more description of the problems might have helped the novice form a better judgment about the significance of a particular problem.

Several recent advances in dimension theory have made certain parts of the book obsolete, particularly the statements of research problems. For example, David Henderson has given an example of an infinite-dimensional metric continuum having no compact positive-dimensional subsets [**3**]. This example appeared while the book was in press and so the author was unable to describe it in detail, although he was able to insert a footnote referring the reader to Henderson's paper. Even more recently the work of V. V. Filippov, B. A. Pasynkov, and I. K. Lifanov has shown the existence of various compact Hausdorff spaces X having the property that $\operatorname{ind} X < \operatorname{Ind} X$. Filippov was the first to give such an example with $\operatorname{ind} X = 2$ and $\operatorname{Ind} X = 3$. His best example [**2**] shows that for each positive integer n there is a compact Hausdorff space X with the property that $\dim X = 1$, $\operatorname{ind} X = n$, and $\operatorname{Ind} X = 2n - 1$. Now for Lindelöf spaces X it is known that $\dim X \leqq \operatorname{ind} X \leqq \operatorname{Ind} X$. Thus Filippov's last example gives a fairly complete answer to the question of how the three principal dimension functions dim, ind, and Ind can vary on compact Hausdorff spaces. The problem of whether ind and Ind can be different for a compact Hausdorff space is stated on p. 123 of the reviewed text. It is unfortunate that these examples appeared shortly after the book was published. It would have been good to have these important examples carefully exposited in the author's lucid style.

Another recent development in dimension theory has been the examples of light open mappings on manifolds constructed by D. C. Wilson. These examples are very important and would have deserved mention if they had appeared sooner. Wilson has shown the existence of light open mappings on manifolds which raise dimension. In particular, he has shown that if $k \geqq 3$ and M^k is a triangulated compact k-manifold and $n \geqq k$, then there is an open mapping $f(M^k) = I^n$ with each point inverse of f homeomorphic to the Cantor set. The work of Wilson has been done since the appearance of the book, but mention should have been made of the problem of dimension raising mappings on manifolds and the work of R. D. Anderson and L. Keldyš in constructing monotone open mappings on n-cells which raise dimension.

In the chapter on dimension-changing closed mappings, there are two results which should have appeared in footnotes. E. G. Skljarenko **[6]** has shown that if X and Y are paracompact spaces and $f(X) = Y$ is a closed mapping, then $\dim X \leqq \dim Y + \dim f$. The corresponding theorem proved in the text in this section is due to K. Morita and states that $\dim X \leqq \operatorname{Ind} Y + \dim f$ under the same conditions on f, X, and Y. However, for any normal space X, $\dim X \leqq \operatorname{Ind} X$ so that Skljarenko's theorem includes Morita's. The proof of Skljarenko's theorem probably could not have been included since it depends on sheaf theory. The other omission in this chapter is a reference to a result of A. V. Zarelua **[8]**. He has shown that if $f(X) = Y$ is a closed mapping which finite-to-one such that $f^{-1}(y)$ consists of at most $k + 1$ points for all y in Y with X and Y normal spaces, then $\dim Y \leqq \dim X + k$. The corresponding theorem proved in the text, $\dim Y \leqq \operatorname{Ind} X + k$, is due to K. Morita and is superseded by Zarelua's theorem. Sheaf theory is also used in the proof of Zarelua's theorem.

There is one example which would have made a worthwhile addition to the book. It would have found easy inclusion in the chapter on the product theorem (or in an appropriate section of the appendix in a footnote). It is due to R. D. Anderson and J. Keisler **[1]**. For each positive integer n, they have shown the existence of a separable metric space X having the property that $\dim X = n$ with X homeomorphic to its own countable infinite product, X^ω. This demonstrates more forcefully than any example in Nagami's text the difficulties involved in showing when equality holds in the product theorem, $\dim (X \times Y) \leqq \dim X + \dim Y$, for noncompact spaces X and Y. Theorem 41–5 states that for a compact space X, $\dim X^2 \geqq 2 \cdot \dim X - 1$. The above example shows the hopelessness of any inequality of this sort for noncompact spaces. The best one can guarantee is that $\dim X^2 \geqq \dim X$ and equality may hold for separable metric spaces of arbitrarily high finite dimension.

Not only are the frontiers of dimension theory expanding, but its foundations and principles are becoming simpler and more elegant. P. Ostrand [5] has recently developed a novel approach to the study of Lebesgue covering dimension which allows one to prove in a simple and elegant fashion many of the classical theorems in dimension theory, including the theorem that $\dim X = \operatorname{Ind} X$ for metric spaces. It seems likely that his approach will ultimately lead to a greatly simplified development for the theory of Lebesgue covering dimension. Despite the elegance and efficiency of Nagami's development I would be disappointed if future works on dimension theory do not make use of Ostrand's ideas to produce an even more exciting and transparent approach to dimension theory.

REFERENCES

1. R. D. Anderson and J. Keisler, *An example in dimension theory*, Proc. Amer. Math. Soc. **18** (1967), 709–713. MR **35** #6130.
2. V. V. Filippov, *On bicompacta with unequal dimensions* ind *and* dim, Dokl. Akad. Nauk SSSR **192** (1970), 516–519 = Soviet Math. Dokl. **11** (1970), 687–691. MR **42** #1082.
3. D. W. Henderson, *An infinite-dimensional compactum with no positive-dimensional compact subsets—a simpler construction*, Amer. J. Math. **89** (1967), 105–121.
4. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N.J., 1941. MR **3**, 312.
5. P. A. Ostrand, *Covering dimension in general spaces*, General Topology and Appl. **1** (1971), 209–221.
6. E. G. Skljarenko, *A theorem on mappings which lower the dimension*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 429–432. (Russian) MR **26** #6933.
7. D. C. Wilson, *Open mappings on manifolds and a counterexample to the Whyburn conjecture* (to appear).
8. A. V. Zarelua, *Finitely-multiple mappings*, Dokl. Akad. Nauk SSSR **172** (1967), 775–778 = Soviet Math. Dokl. **8** (1967), 172–175. MR **35** #3640.

JAMES KEESLING

Diffusion Processes and their Sample Paths by Kiyosi Ito and Henry P. McKean, Jr., Springer-Verlag, Berlin, 1965.

This book is an excellent illustration of the thesis that once the foundation for solving a basic problem is laid, no matter how complicated the solution may be the problem will find authors equal to the task. In the present case, the historical genesis of the problem goes back to the work of W. Feller in the early 1950's, on characterizing the most general diffusion operator in one dimension. In this work the role of probability was largely confined to motivation and the aim was to characterize certain differential operators by purely analytic axioms. At the time when Feller's treatment of the subject reached its most final form (Illinois J. Math. 1957, 1958) one finds the issue stated as that