BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

THE SPECTRAL MAPPING THEOREM IN SEVERAL VARIABLES

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Communicated by Robert G. Bartle, February 11, 1972

1. **Introduction.** In this note we introduce a "joint spectrum" for systems of Banach algebra elements, one which reduces to the classical "spectrum" for a single element, and which is subject to the "spectral mapping theorem for polynomials" when the elements of the system commute with one another. In contrast to the known result for a commutative algebra, which we generalise, the proof is comparatively elementary: we need only the argument from Liouville's theorem which says that the spectrum of a bounded linear operator on a nontrivial Banach space is not empty.

2. **The joint spectrum.** If A is a complex linear algebra, with identity 1, and if $a = (a_1, a_2, \ldots, a_n)$ is an n-tuple of elements of A, then the *joint spectrum* of a with respect to A is to be the set of n-tuples $s = (s_1, s_2, \ldots, s_n)$ of complex numbers for which the system $a - s = (a_1 - s_1, a_2 - s_2, \ldots, a_n - s_n)$ generates a proper left or right ideal in A. Formally

$$\sigma(a) = \sigma_A^{\text{joint}}(a) = \sigma_A^{\text{left}}(a) \cup \sigma_A^{\text{right}}(a) \tag{2.1}$$

is the union of the "left" and "right" spectrum, where

$$\sigma_A^{\text{left}}(a) = \left\{ s \in C^n : 1 \notin \sum_j A(a_j - s_j) \right\} \tag{2.2}$$

and

$$\sigma_A^{\text{right}}(a) = \left\{ s \in C^n : 1 \notin \sum_j (a_j - s_j)A \right\}. \tag{2.3}$$

For example, if the algebra A is a commutative Banach algebra, it is the Gelfand theory that, if $a = (a_1, a_2, \ldots, a_n)$,

$$\sigma_A^{\text{left}}(a) = \sigma_A^{\text{right}}(a) = \{ (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)) : \varphi \in \Phi \}, \tag{2.4}$$

where Φ is the "maximal ideal space" of A: this is the usual definition for a commutative algebra [3]. If A is the algebra of bounded linear operators on a complex Hilbert space E, and $T = (T_1, T_2, \ldots, T_n)$, then it turns out ([2], [3], [4]) that

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AMS 1970 *subject classifications.* Primary 47D99, 46H99; Secondary 47A10, 47A60.
Key words and phrases. Joint spectrum, spectral mapping theorem, Banach algebras, polynomial in several variables, normal operator on Hilbert space.

$$\sigma_A^{\text{left}}(T) = \left\{ s \in C^n : \inf_{\|x\|=1} \sum_j \|(T_j - s_j)x\| = 0 \right\} \tag{2.5}$$

is the set of "simultaneous approximate eigenvalues" of the system, while

$$\sigma_A^{\text{right}}(T) = \left\{ s \in C^n : \sum_j (T_j - s_j)E \neq E \right\}. \tag{2.6}$$

In general the joint spectrum $\sigma(a)$ of an arbitrary system $a \in A^n$ of Banach algebra elements is a compact subset of C^n: for the inclusion

$$\sigma(a) \subseteq \sigma(a_1) \times \sigma(a_2) \times \ldots \times \sigma(a_n) \tag{2.7}$$

guarantees that it is bounded; to see that it is also closed, suppose that $s \in C^n$ is not in $\sigma(a)$. There are therefore systems $a' \in A^n$ and $a'' \in A^n$ for which

$$\sum_j a_j'(a_j - s_j) = \sum_j (a_j - s_j)a_j'' = 1; \tag{2.8}$$

now if the system $s' \in C^n$ is so close to s in C^n that

$$\sum_j \|a_j'\| \, |s_j' - s_j| < 1 \quad \text{and} \quad \sum_j \|a_j''\| \, |s_j' - s_j| < 1$$

then each of the elements $\sum_j a_j'(a_j - s_j')$ and $\sum_j (a_j - s_j')a_j''$ is invertible in A, which excludes s' from $\sigma(a)$.

Even in the very simplest situations, the joint spectrum is liable to be empty: in the algebra A of complex 2×2 matrices take $a = (a_1, a_2)$ with

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.9}$$

It is clear from (2.7) that the only possible point of the spectrum is (0, 0): but $a_2a_1 + a_1a_2$ is the identity matrix.

3. **The spectral mapping theorem.** In a complex linear algebra A, for an arbitrary system $a = (a_1, a_2, \ldots, a_n)$ of elements and a system $f: A^n \to A^m$ of "polynomials in n variables on A," there is inclusion

$$f\sigma(a) \subseteq \sigma f(a). \tag{3.1}$$

To interpret this, we shall regard a "polynomial" as one of the mappings from A^n into A in the linear algebra generated by the scalar *constants* s: $(a_1, a_2, \ldots, a_n) \to s$ and the *co-ordinates* $z_j: (a_1, a_2, \ldots, a_n) \to a_j$ $(j = 1, 2, \ldots, n)$. Systems $f = (f_1, f_2, \ldots, f_m)$ of these polynomials are identified with the corresponding mappings of A^n into A^m; restricted to the scalar systems $C^n \subseteq A^m$, such a mapping then reduces to a system of numerical polynomials.

The proof of the "one-way spectral mapping theorem" (3.1) is a matter of the "remainder theorem" for the polynomials $f_k: A^n \to A$: if $a \in A^n$ and $s \in C^n$ are arbitrary then

$$(3.2) \quad f_k(a) - f_k(s) \in \sum_j A(a_j - s_j) \quad \text{and} \quad f_k(a) - f_k(s) \in \sum_j (a_j - s_j)A.$$

This is easily built up for sums of products of constants and co-ordinates. Now argue that if $s \in C^n$ and $f(s)$ is not in $\sigma f(a)$, s cannot be in $\sigma(a)$.

In general we cannot expect equality in (3.1): indeed as in (2.9) whenever the spectrum of a system $a \in A^n$ is empty, and $f = f_1 : A^n \to A$ is one polynomial, then

$$(3.3) \qquad f\sigma(a) = \varnothing \neq \sigma f(a).$$

The argument in the other direction, for commuting systems of Banach algebra elements, is extracted from Bunce [**2**, Proposition 1]:

LEMMA. *In a complex Banach algebra A with identity, suppose that $b \in A^m$ is an arbitrary system of elements, and that $a \in A^n$ is a commuting system of elements commuting with the elements of b. Then for every point t in $\sigma(b)$ there is $s \in C^n$ for which (s, t) is in $\sigma(a, b)$.*

PROOF. We carry out the argument for the "left" spectrum (2.2), in the case $n = 1$ of a single element $a = a_1$ commuting with the system $b \in A^m$. If $t \in C^m$ is in $\sigma_A^{\text{left}}(b)$ then the closed left ideal

$$(3.4) \qquad N = \text{closure} \sum_k A(b_k - t_k)$$

is proper in A: consider the closed subspace

$$(3.5) \qquad M = \{c = c_1 \in A : (b_k - t_k)c \in N \ (k = 1, 2, \ldots, m)\}.$$

The subspace M contains the ideal N, as well as the element 1 which is excluded from N: thus the quotient space M/N is nontrivial. While M is not in general an ideal, there is an inclusion $aM \subseteq M$ whenever $a = a_1$ commutes with each of the elements b_k. For suppose $c \in M$. Then, for each $k = 1, 2, \ldots, m$,

$$(b_k - t_k)ac = a(b_k - t_k)c \in aN \subseteq N.$$

Consider the operator $L_a: c + N \to ac + N$ $(M/N \to M/N)$. By the usual argument from Liouville's theorem [**5**, Theorem 67A] the spectrum of L_a is not empty. Further, for every element $s = s_1$ of the topological boundary of this spectrum, the operator $L_a - sI$ has an "approximate eigenvector" in the space M/N [**5**, Theorem 66B]:

$$(3.6) \qquad \inf_{\|c+N\|=1} \|(a - s)c + N\| = 0.$$

The usual argument actually furnishes a sequence (T_n) of operators on M/N for which $\|T_n\| = 1$ and $\|(L_a - s)T_n\| \to 0$: but these are readily converted to vectors $c_n + N$. We claim that for every such point s, the $(m+1)$-tuple (s, t) is in the left spectrum $\sigma_A^{\text{left}}(a, b)$: indeed if, to the contrary, there are a' and b'_k for which $a'(a - s) + \sum_k b'_k(b_k - t_k) = 1$ then, for arbitrary c in M,

$$c = a'(a - s)c + \sum_k b'_k(b_k - t_k)c \in a'(a - s)c + N.$$

It follows that $\|c + N\| \leqq \|a'\| \; \|(a - s)c + N\|$, which is incompatible with (3.6).

The "spectral mapping theorem" follows at once:

THEOREM. *If $a \in A^n$ is a commuting system of elements of a complex Banach algebra A with identity, and if $f: A^n \to A^m$ is a system of polynomials, then the spectrum $\sigma(a)$ is not empty, and there is equality $\sigma f(a) = f\sigma(a)$.*

PROOF. With $b = f(a) \in A^m$ it is the Lemma that for each point t in $\sigma(b)$ there is $s \in C^n$ for which $(s, t) \in C^{n+m}$ is in $\sigma(a, f(a))$. By a trivial application of (3.1) it is clear that s must be in $\sigma(a)$; we claim that also $t = f(s)$. For consider the system of polynomials $g: (a, b) \to b - f(a)$ $(A^{n+m} \to A^m)$. By (3.1),

$$t - f(s) = g(s, t) \in g\sigma(a, f(a)) \subseteq \sigma g(a, f(a)) = \sigma(0) = \{0\}.$$

This is the second part of the statement; the first is obvious, and can be derived from (3.3).

4. **Normal operators.** If $T = (T_1, T_2, \ldots, T_n)$ is a commuting system of bounded linear operators on a Banach space then the spectral mapping theorem, with (2.7), gives inclusion

$$\sigma f(T) \subseteq f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_n)); \tag{4.1}$$

just occasionally this may help in the computation of the spectrum of a particular operator. For example if T_2 is quasinilpotent and commutes with T_1 it is a consequence of (4.1) that

$$\sigma(T_1 + T_2) \subseteq \sigma(T_1) \subseteq \sigma(T_1 + T_2). \tag{4.2}$$

For another consequence, suppose that $T = T_1$ is a normal operator on complex Hilbert space: then

$$\|T\| = \sup\{|s| : s \in \sigma(T)\}. \tag{4.3}$$

While the proof of the Lemma has in a sense [1] ruled us "out of court" here, the argument is not without interest. It is clear that $\|T\|^2 \in \sigma(T^*T)$: for if $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|$ then $\|(T^*T - \|T\|^2)x_n\|^2 \to 0$ Conclude

$$\|T\|^2 = \|T^*T\| \leqq \sup_{s,t \in \sigma(T)} |s|\,|t^*| = \sup_{s \in \sigma(T)} |s|^2.$$

We have used the fact that the spectrum of T^* is the complex conjugate of the spectrum of T, and applied (4.1) to the product T^*T. If we establish further the inclusion

$$\sigma(T, T^*) \subseteq \{(s, s^*): s \in \sigma(T)\} \tag{4.4}$$

(whether or not T is normal, if $t \neq s^*$ then $(T^* - s^*)(T - sI) + (T - t^*)(T^* - tI)$ is of the form $S^*S + SS^* + kI$ with $k > 0$, and therefore invertible), then the spectral mapping theorem gives

$$\begin{aligned} \sigma f(T, T^*) &= \{f(s, s^*): s \in \sigma(T)\}, \\ \|f(T, T^*)\| &= \sup\{|f(s, s^*)|: s \in \sigma(T)\}, \end{aligned} \tag{4.5}$$

hence the "spectral theorem" for T [**1**].

Note that it is necessary, for (4.5), that the operator T be normal: for then the hermitian operator $i(T^*T - TT^*)$ has spectrum $\{0\}$ and must be 0.

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