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# ON THE HAEFLIGER KNOT GROUPS

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ABSTRACT. Let $C_n^k$ be the group of isotopy classes of differentiable embeddings $S^n \subset S^{n+k}$. In [4], A. Haefliger established an isomorphism $C_n^k \cong \pi_{n+1}(G,SO, G_k)$ where $G_k$ is the set of (oriented) homotopy equivalences of the sphere $S^{k-1}$. In this note, we indicate methods which make the calculation of these groups feasible. In particular, we determine the first seven nonzero groups in the metastable range. We also develop connections between the composition operation in homotopy and such geometric operations as spin-twisting knots.

## 1. The space of the Haefliger knot groups.

THEOREM A. *There is a space $F_k$ which is $2k-3$-connected and $\pi_n(F_k) \cong C_n^k$.*

Indeed, $F_k$ is the fiber in the map

$$SG_k/SO_k \to SG/SO,$$

induced by the usual inclusions $B_{G_k} \hookrightarrow B_G$, $B_{SO_k} \hookrightarrow B_{SO}$. Alternately, $F_k$ is the fiber in the map

$$SO/SO_k \to G/G_k,$$

induced by the inclusions $B_{SO_k} \hookrightarrow B_{G_k}$, $B_{SO} \hookrightarrow B_G$.

COROLLARY B. *$H_*(F_k; Z_2) \cong E(\cdots A_I \cdots)$ where $I$ runs over all sequences of integers $(i_1, \ldots, i_t)$, satisfying*

(i) $0 \leqq i_1 \leqq \ldots \leqq i_t$ $(t \geqq 2)$,
(ii) $i_1 = 0$ *implies* $t = 2$,
(iii) $i_t \geqq k-1$.

*Moreover,* $\dim(A_I)$ *is* $i_1 + 2i_2 + 4i_3 + \ldots + 2^{t-1}i_t - 1$. *(Here $E$ is an exterior algebra on these stated generators.)*

B follows from A on applying the results of [7]. In the same way, it is possible to obtain partial information about $H_*(F_k; Z_p)$ for $p$ odd. Similarly, we can determine $H^*(F_k; Z_2)$ as a module over the Steenrod algebra $\mathscr{A}(2)$, and $H^*(F_k; Z_p)$ over $\mathscr{A}(p)$ in the range of dimensions less than $3k - 2$.

In this range, $H^*(F_k; Z_p)$ has one nonzero generator $e_{4s-1}$ in each dimension $4s-1$, and is zero otherwise. For general $p$, the $\mathscr{A}(p)$-structure

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is involved. However, for $p = 3$ we have the concise formula

$$\mathscr{P}^i(e_{4s-1}) = \binom{2s}{i}(e_{4s-1+4i}).$$

The structure of $H^*(F_k; Z_2)$ is considerably more complex.

COROLLARY C. *In dimensions less than* $3k-2$, $H^*(F_k; Z_2)$ *has generators of the form*

$$e^i \cup (e^j \otimes e^j),$$

*with* $j \geqq k-1$, *of dimension* $2j + i - 1$ *and*

$$Sq^r(e^i \cup e^j \otimes e^j) = \sum_s \binom{i+j-s}{r-2s}\binom{j}{s} e^{i+r-2s} \cup e^{j+r} \otimes e^{j+s}.$$

(Compare [8, §3] where a space very similar to this is studied.)

COROLLARY D. $(F_{k+1} \cup cF_k) \simeq \sum^{k+1} P_{k-1}$ *in our range.*

Here $P_{k-1}$ is the truncated projective space $P^\infty/P^{k-2}$. Also, the map $F_k \to F_{k+1}$ is the natural one which geometrically corresponds to compositions $S^k \subset S^{n+k} \subset S^{n+k+1}$. D gives a homotopy-theoretical proof of the second main result of [4], namely, the isomorphism

$$\pi_n(F_q, G_q) \cong \pi_{n-q+1}(SO, SO_{q-1})$$

for $n \leqq 3q-6$.

Finally, passing to rational homology, the map $H_*(\Omega(G/SO)) \to H_*(F_k)$ is surjective, and we have

COROLLARY E.

$$\pi_i(F_k) \otimes Q = \begin{cases} Q, & i \equiv 3(4), \quad i > 2k-3, \\ 0 & \textit{otherwise}. \end{cases}$$

*In particular, for* $i \equiv 3(4)$ *and* $i > 2k-3$, $\pi_i(F_k)$ *contains a Z-direct summand.*

The identification of the Hurewicz image of these summands is a major problem in further clarifying the homotopy type of the $F_k$. Some results should be possible using the techniques of [3].

2. **Some calculations.** We use Corollary C to calculate the $E^2$-term of the Adams spectral sequence for $F_k$ in dimensions less than $3k-3$ at the prime two. (In the range in which the calculations are carried through, the $p$-primary calculations $p = 3, 5, 7 \ldots$ are direct.) The $E^2$-terms for $p = 2$ exhibit a type of periodicity

$$\mathrm{Ext}^{s,t}_{\mathscr{A}(2)}(H^*(F_{k+2^r}), Z_2) \cong \mathrm{Ext}^{s,t-2^{r+1}}_{\mathscr{A}(2)}(H^*(F_k), Z_2)$$

for $t - s < 2^r + 2(k + 2^r - 1)$. The author does not know if this periodicity has any geometric analogue, however.

These Ext groups are calculated by use of the tables in [6] and the spectral sequence techniques in [8, §9]. After evaluating all the differentials we can obtain

THEOREM F. *The first seven two-primary components of the $C_n^k$ for $k > 7$ are*

(a) *for* $k \equiv 2(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z_2$ | 0 | $Z \oplus Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z \oplus (Z_2)^2$; |

(b) *for* $k \equiv 3(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z$ | $Z_4$ | $Z_2$ | 0 | $Z \oplus Z_2$ | $Z_2 \oplus Z_{16}$ *or*<br>$Z_2 \oplus Z_8$ | $Z_4 \oplus Z_2$ *or*<br>$Z_2 \oplus Z_2$ *or*; |

(c) *for* $k \equiv 4(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z_2$ | 0 | $Z$ | $Z_2$ | $Z_2 \oplus Z_2$ | $Z_8 \oplus Z_4$ | $Z \oplus Z_2$ | $Z_2$; |

(d) *for* $k \equiv 5(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z$ | $(Z_2)^2$ | $(Z_2)^3$ | $Z_{16} \oplus Z_8$ *or*<br>$Z_{16} \oplus Z_4$ | $Z \oplus (Z_2)^2$ | $Z_2$ | $Z_2$; |

(e) *for* $k \equiv 6(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z_2$ | 0 | $Z \oplus Z_2$ | $Z_2$ | $Z_2$ | 0 | $Z \oplus Z_2$; |

(f) *for* $k \equiv 7(8)$,

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C^k_{2k-3+j}$ | $Z$ | $Z_4$ | $Z_2$ | 0 | $Z$ | $Z_8$ *or*<br>$Z_4$ *or*<br>$Z_2$ | $Z_2 \oplus Z_4$ *or*<br>$Z_2 \oplus Z_2$ *or*<br>$Z_2$; |

(g) *for* $k \equiv 0(8)$, *the groups are isomorphic to those for* $k \equiv 4(8)$;

(h) *for* $k \equiv 1(8)$, *the groups are isomorphic to those for* $k \equiv 5(8)$, *except possibly for* $j = 3$, *where, however, the same two groups are the only possibilites.*

Taking into account the 3-primary structure, we obtain

COROLLARY G. *$C_n^k = 0$ for $n < 2k + 4$ if and only if*
(a) $k \equiv 0(2)$ *and* $n = 2k-2$,
(b) $k \equiv 3(4)$, $k \not\equiv 1(3)$, *and* $n = 2k$,
(c) $k \equiv 6(8)$, $k \not\equiv 0(3)$, *and* $n = 2k + 2$.

Thus, in these dimensions, a smooth embedding

$$f: M^n - (\text{point}) \subset S^{n+k}$$

implies the existence of a smooth embedding

$$\tilde{f}: M^n \subset S^{n+k+1}.$$

REMARK. The ambiguities in $F$ for $k \equiv 5(8)$ and $j = 3$ would be resolved if we knew the image of the Hurewicz map in dimension $2k$.

3. **Some operations on the $C_n^k$.** The map $C_n^k \mapsto C_n^{k+1}$ is obtained by passing to homotopy in the map $h: F_k \to F_{k+1}$.

PROPOSITION H. *$h^*$ is injective on $H^i(F_{k+1}; Z_p)$ for $i < 3k-2$. In particular,*

$$h^*(e^r \cup e^j \otimes e^j)_{k+1} = (e^r \cup e^j \otimes e^j)_k.$$

This enables us to obtain information on how far back a particular embedding desuspends. For example, let $g$ be the generator [3] of $C_{2k-3}^k$ with $k$ odd. Then

THEOREM I. (a) *If $k \equiv 1(4)$, $g$ is not the suspension of any knot $S^{2k-3} \hookrightarrow S^{3k-4}$.*

(b) *If $k \equiv 3(4)$, $g$ is the suspension of a knot $\bar{g}: S^{2k-3} \hookrightarrow S^{3k-4}$ but is not the double suspension of any knot $S^{2k-4} \hookrightarrow S^{3k-5}$.*

Composition defines an action of $\pi_*^S(S^0)$, the stable homotopy of spheres, on the $C_n^k$ for $k > \frac{1}{2}n$. For example, if $\eta \in \pi_1(S^0) = Z_2$ represents the nontrivial element, then $\eta_0$ constructs from a knot $f: S^n \hookrightarrow S^{n+k}$ a new knot

$$f_\eta: S^{n+1} \hookrightarrow S^{n+k+1}.$$

On the other hand, Artin-Zeeman [2], [9] defined the notion of twist-spinning knots, and Hsiang-Sanderson [5] generalized the twist-spinning construction considerably.

THEOREM J. (a) *Composition with $\eta$ corresponds to the Artin-Zeeman twist-spinning operation.*

(b) *Composition with $J(\alpha)$ corresponds to the Hsiang-Sanderson construction for $\gamma = [1, \alpha]$. (Here, $J: \pi_*(SO) \to \pi_*^S(S^0)$ is the Whitehead J-homomorphism [1].)*

COROLLARY K. (a) *Iterating the Artin-Zeeman twist-spinning operation four times* (*for* $k > 2$) *always results in a trivial knot.*

(b) *The Artin-Zeeman twist-spinning of* $g_k$ *is never zero. The iteration is nonzero if and only if* $k \equiv 1(4)$.

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