BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

ON $\operatorname{Ext}_R^1(A, R)$ FOR TORSION-FREE A

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Communicated by Joseph Rotman, March 13, 1972

For any Dedekind domain R (or more generally a Prüfer domain) the modules of the form $\operatorname{Ext}_R^1(A, R)$, A being a torsion-free R-module, coincide with those of the form $\varprojlim^{(1)} F_\alpha$ for projective systems of finitely generated free R-modules F_α (cf. [1], [5]) and appear in various topological contexts.

It is the purpose of this note to outline some results concerning the structure of the above class of modules, in particular, in the case where A is the quotient field Q of R. As a by-product we get a negative solution of an analogon of the following (open) problem [4]: If Z is the ring of integers, can $\operatorname{Ext}_Z^1(A, Z)$ ever be a nonzero torsion group? In fact, we shall obtain a class of principal ideal domains R such that $\operatorname{Ext}_R^1(A, R) \simeq Q/R$ for a suitable torsion-free R-module A.

Generally, for any integral domain R with quotient field Q $\operatorname{Ext}_R^1(Q, R)$ may be viewed as a vector space over Q and thus

$$(*) \qquad \operatorname{Ext}_R^1(Q, R) \simeq Q^{(d)},$$

for some finite or infinite cardinal number d.

THEOREM 1. *Let R run through the class of all Noetherian domains of Krull dimension* 1 *which are analytically unramified in at least one maximal ideal. Then the cardinal numbers d which appear in* (*) *are*

(i) *any infinite cardinal number,*

(ii) *among the finite cardinal numbers exactly those of the form $p^t - 1$, p being a prime number and t an integer $\geqq 0$.*

When d is finite, and $\neq 0$ R is necessarily local and of prime characteristic. Moreover, any d from (i) *and* (ii) *can occur for principal ideal domains, in fact even for valuation rings.*

Before we sketch the main steps in the proof we show how we by Theorem 1 (for $p = 2$, $t = 1$) get a principal ideal domain R such that $\operatorname{Ext}_R^1(A, R) \simeq Q/R$ for some torsion-free R-module A.

PROPOSITION 2. *If R is a discrete valuation ring for which $\operatorname{Ext}_R^1(Q, R) \simeq Q$, then there exists a torsion-free R-module A such that $\operatorname{Ext}_R^1(A, R) \simeq Q/R$.*

PROOF. Since $\operatorname{Ext}_R^1(Q, R) \neq 0$, R is not a complete valuation ring. By [2, Theorem 19], there exists an indecomposable torsion-free R-module A

AMS 1970 *subject classifications.* Primary 20K35, 18G15; Secondary 16A62.

of rank 2. A contains a cyclic pure submodule for which the corresponding quotient is Q, i.e. there is an exact sequence

$$(**) \qquad 0 \to R \to A \to Q \to 0.$$

Here $\mathrm{Hom}_R(A, R) = 0$, since otherwise R would be a homomorphic image of A, contradicting the indecomposability of A. Hence (**) gives rise to an exact sequence

$$0 = \mathrm{Hom}_R(A, R) \to R \to \mathrm{Ext}^1_R(Q, R) \to \mathrm{Ext}^1_R(A, R) \to 0,$$

from which we conclude that $\mathrm{Ext}^1_R(A, R) \simeq Q/R$.

For the proof of Theorem 1 we first show that any d from (i) or (ii) actually appears in (*) for a suitable principal ideal domain. We use the following class of rings, similar to some of Nagata's "bad rings". For a prime number p let F_p denote the field with p elements and consider $K = F_p(\{X_\alpha\})$, $\alpha \in I$, the field of rational functions in X_α, $\alpha \in I$, where the index set I has cardinality $\max(\aleph_0, d)$. Let $S = K[[Y]]$ be the ring of formal power series over K and L the corresponding quotient field. If L' is the subfield generated by L^p, $\{x_\alpha\}$, $\alpha \in I$, and Y, the p-dimension (in the sense of Teichmüller) of L with respect to L' is $\geqq \max(\aleph_0, d)$. Consequently there exists a subfield $Q \subset L$ such that $[L:Q] = d$ for d infinite or $[L:Q] = p^t$, t an integer $\geqq 0$. $R = S \cap Q$ is a discrete valuation ring having S as its completion. By standard arguments one infers that $\mathrm{Ext}^1_R(Q, R) \simeq Q^{(d)}$ when d is infinite and $\mathrm{Ext}^1_R(Q, R) \simeq Q^{(p^t - 1)}$ in the finite case.

For the proof of the other statements in Theorem 1 (and possible generalizations) it is convenient to use henselizations. Reduction to the local case is done by the following:

Proposition 3. *Let R be an arbitrary Noetherian domain. If the localization $R_\mathfrak{m}$ is henselian for some maximal ideal $\mathfrak{m}$, then R is a local ring.*

The next proposition gives a sufficient condition for d in (*) to be infinite.

Proposition 4. *Let R be a one-dimensional Noetherian domain. Then d in (*) is infinite if either R is nonlocal or R is a local analytically irreducible ring for which the residue field is perfect.*

By use of Proposition 4 and generalizations of standard methods from abelian groups one gets

Theorem 5. *Let R be one-dimensional Gorenstein domain satisfying one of the following conditions:*

(i) Char $R = 0$, *and* R *analytically unramified in at least one maximal ideal.*

(ii) R *is nonlocal, and* R *analytically unramified in at least one maximal ideal.*

(iii) R *is a local analytically unramified ring for which the residue field is perfect.*

If A *is a countably generated torsion-free* R*-module, then* $\text{Ext}_R^1(A, R)$ *is never a nonzero torsion module.*

Let us in passing note an application of Theorem 1 concerning certain chain conditions. Recall that ACC_n for a module means that the n-generated submodules satisfy the ascending chain condition. Using an idea of G. Bergman and a result in [3] one gets

PROPOSITION 6. *The following are equivalent for a local Noetherian one-dimensional analytically unramified domain* R:

(i) R *is complete.*

(ii) *For torsion-free* R*-modules* $ACC_n \Rightarrow ACC_{n+1}$ *for at least one* n*, where* $n < p$ *if* R *has characteristic* $p > 0$.

Finally we look at Prüfer domains of Krull dimension 1.

THEOREM 7. *For one-dimensional Prüfer domains* R *(in fact even for Bezout domains) any* d *can occur in* (*). *When* R *runs through the one-dimensional Prüfer domains, whose quotient fields are not algebraically closed the finite* d*'s which occur in* (*) *are exactly the odd numbers and the numbers of the form* $p^t - 1$, p *being a prime number and* t *an integer* $\geqq 0$.

COROLLARY. *There exists a nondiscrete valuation ring* R *of rank one and characteristic zero such that* $\text{Ext}_R^1(A, R) \simeq Q/R$ *for some torsion-free* R*-module* A.

The situation seems to be more complicated for higher dimensional Noetherian domains, where one is led to consider the R-topology. It might be of interest to know what would be the analogue of Theorem 1 for arbitrary Noetherian domains. We conclude by mentioning the following proposition for polynomial rings, (obtained by "globalization" of a result of L. Gruson):

PROPOSITION 8. *Let* F *be a field and* $R = F[X_1, \ldots, X_n]$ *the polynomial ring in* n *indeterminates. If* F *is at most countable or* $n = 1$, *then* d *in* (*) *is* $2^{|R|}$. *Otherwise* $d = 0$.

In particular, Proposition 8 gives a large class of nonlocal Noetherian domains (with trivial Jacobson radical) which are complete in the R-topology.

References

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