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INVOLUTIONS ON KLEINIAN GROUPS

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The purpose of this note is to give an elementary proof of the following result.

THEOREM A. *Let G be a finitely generated nonelementary Kleinian group and let J be an anticonformal homeomorphism of $\Omega = \Omega(G)$, the set of discontinuity of G, where J commutes with every element of G. Then J is the restriction of an anticonformal, involutory fractional linear transformation (that is, $J(z) = (a\bar{z} + b)/(c\bar{z} + d)$, $J^2 = 1$) and G is either Fuchsian or a Z_2-extension of a Fuchsian group. Further, the mapping J with the above properties is unique.*

We prove Theorem A by reducing it to

THEOREM B. *Let Γ be a finitely generated Fuchsian group operating on U_1 and U_2, the upper and lower half-planes, respectively. Let f_1 and f_2 be schlicht functions on U_1 and U_2, where $f_1 \circ \gamma \circ f_1^{-1}$ and $f_2 \circ \gamma \circ f_2^{-1}$ both define the same isomorphism of Γ onto a Kleinian group G, and $f_1 = f_2$ on that part of the real axis $\boldsymbol{R}$ lying in $\Omega(\Gamma)$. Then f_1 and f_2 are restrictions of the same fractional linear transformation.*

As a corollary to our proof of Theorem B, we obtain the somewhat more general

THEOREM C. *Let Γ be a finitely generated Fuchsian group of the first kind acting on U_1 and U_2. Let f_1 defined on U_1, and f_2 defined on U_2 be holomorphic cover mappings where $f_1 \circ \gamma \circ f_1^{-1}$ and $f_2 \circ \gamma \circ f_2^{-1}$ both define the same homomorphism of Γ onto a Kleinian group G. Then G is either Fuchsian or a Z_2-extension of a Fuchsian group (perhaps of the second kind).*

REMARK. Theorem C gives information about certain deformations of Γ, in the sense of Kra [**6**], where the same deformation is supported in both U_1 and U_2. Nothing is known about the more general case where f_1 and f_2 are merely locally schlicht.

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Using standard techniques in quasiconformal mappings, we also get an elementary proof of the following result of Maskit [9].

THEOREM D. *Let G be a finitely-generated Kleinian group with two invariant components. Then G is a quasiconformal deformation of a Fuchsian group.*

We start by giving a

PROOF OF THEOREM B. We denote the map $z \mapsto \bar{z}$ by j. Note that we have a well defined mapping $f\colon \Omega(\Gamma) \to \Omega(G)$; *a priori* this mapping need not be surjective. It projects to a surjective mapping-display $f^*\colon \Omega(\Gamma)/\Gamma \to f(\Omega(\Gamma))/G$. Since $\Omega(\Gamma)/\Gamma$ is of finite type, $f(\Omega(\Gamma))$ is a union of components of $\Omega(G)$ and f^* is an n-sheeted covering for some $n \geqq 1$. Furthermore, $f^* \mid (U_i/\Gamma)$ is injective for $i = 1, 2$. Thus $n = 1$ or $n = 2$.

If $f_1(U_1) \cap f_2(U_2) = \emptyset$, then G has two invariant connected open subsets of its region of discontinuity: namely $f_1(U_1)$ and $f_2(U_2)$. Thus every noninvariant component of $\Omega(G)$ is an atom (Accola [1]). Since finitely generated Kleinian groups do not have atoms (Ahlfors [2]) we conclude that

$$\Omega(G) = f_1(U_1) \cup f_2(U_2) \cup f_1(\Omega(\Gamma) \cap \boldsymbol{R}).$$

Thus if $f_1(U_1) \cap f_2(U_2) = \emptyset$, we set

$$\begin{aligned} J(z) &= f_2 \circ j \circ f_1^{-1}(z), & z &\in f_1(U_1), \\ &= f_1 \circ j \circ f_2^{-1}(z), & z &\in f_2(U_2), \\ &= z, & z &\notin f_1(U_1) \cup f_2(U_2), \end{aligned}$$

and observe that $J^{-1} \circ g \circ J = g$ for every $g \in G$.

Since f_1 and f_2 both are equivalent (under the Möbius group) to bounded holomorphic functions, by Fatou's Theorem they have locally L_1 (even L_∞) vertical boundary values. Using the Cauchy integral formula it suffices to show that these are the same a.e. Observe that by Maskit [10], J is a homeomorphism. Now if $f_1(w)$ has a limit as $\operatorname{Im} w \to 0$, then either w approaches a point in $\Omega(\Gamma)$, in which case by hypothesis, $f_2(jw)$ tends to the same point as $f_1(w)$; or, since w is schlicht, $f_1(w)$ tends to a point of $\Lambda(G)$, the limit set of G. If $f_1(w)$ tends to a point of $\Lambda(G)$, then $f_2(jw) = J(f_1(w))$ tends to the same point.

If $f_1(U_1) \cap f_2(U_2) \neq \emptyset$, then observe that $f_1(U_1)$ is bounded by the limit points of G and the points in the image of $\Omega(\Gamma) \cap \boldsymbol{R}$, and so $f_1(U_1) = f_2(U_2)$. Then $f_2^{-1} \circ f_1$ is directly conformal, maps U_1 onto U_2, and is the identity on $\Omega(\Gamma) \cap \boldsymbol{R}$ and on the hyperbolic fixed points. Hence, $f_1(U_1) \cap f_2(U_2) \neq \emptyset$ cannot occur.

REMARK. A more direct proof of Theorem B can be obtained by showing that $|f_1(z) - f_2(jz)|$ tends uniformly to zero as $\operatorname{Im} z \to 0$ with $z \in U_1$. This involves an analysis similar to the one appearing in Maskit [10].

PROOF OF THEOREM A. We first observe that J^2 is conformal and commutes with every element of G. Hence (Kra [7] or Maskit [10]), $J^2 = 1$.

Suppose there is a component Δ_1 of G with $\Delta_2 = J\Delta_1 \neq \Delta_1$. Let H be the subgroup of G keeping Δ_1 invariant; obviously $H\Delta_2 = \Delta_2$. By Accola's remark [1], Δ_1 and Δ_2 are both simply-connected. Choose a Fuchsian group Γ and a conformal map $f_1 : U_1 \to \Delta_1$ which conjugates Γ into G. Define $f_2 : U_2 \to \Delta_2$ by $f_2 = J \circ f_1 \circ j$. Since Γ is of the first kind, by Theorem B, f_1 and f_2 are restrictions of the same fractional linear transformation f. Then in Δ_2, $J = f \circ j \circ f^{-1}$; and so $J = f \circ j \circ f^{-1}$ everywhere.

We now assume that J keeps every component of G invariant. Then G has only one component Δ, for set

$$\begin{aligned} J^*(z) &= Jz, && z \in \Delta, \\ &= z, && z \notin \Delta, \end{aligned}$$

and observe that, by Maskit [10], J^* is a global homeomorphism which reverses orientation in Δ, and preserves orientation in the interior of the complement of Δ.

Let Γ be a Fuchsian group, operating on U_1, where $f_1 : U_1 \to \Delta$ is the universal covering, and Γ is the lifting of G; that is, $U_1/\Gamma \cong \Delta/G$. Let Γ^* be the $\boldsymbol{Z}_2$-extension of Γ which covers $G \cup J$.

Suppose that no orientation-reversing $\gamma^* \in \Gamma^*$ had a fixed (non-Euclidean) line in U_1. Then for every such γ^*, $(\gamma^*)^2 = \gamma \in \Gamma$, and A_γ, the axis of γ is invariant under γ^*. Choose γ_0^* to minimize the non-Euclidean length of A_γ/Γ. Then since γ^* projects onto an involution, A_{γ_0}/Γ can have at most one double point. One double point would lift to a fixed point of some γ^*. Hence A_{γ_0}/Γ is a simple loop. Since $J^2 = 1$, A_{γ_0} projects onto a simple loop in Δ. This simple loop is invariant under J; hence J interchanges the two topological discs bounded by the loop. Finally, since J is the identity on $\Lambda(G)$, $\Lambda(G) = \varnothing$ — contradicting the assumption that G is nonelementary. We conclude that some lifting γ^* of J has a line of fixed points in U_1; hence J has fixed points in Δ.

The set T of fixed points of J must divide Δ into at least two regions, for if not, we could repeat the above argument looking at the universal covering of $\Delta - T$. If there were more than two regions, we could as above define $J^* = J$ in two of these regions, and $J^* = 1$ elsewhere, to get a contradiction. Let Δ_1 and Δ_2 be the components of $\Delta - T$. Since

$gT = T$ for all $g \in G$, H, the subgroup of G keeping Δ_1 invariant is of index at most 2 in G. The group H has invariant open sets Δ_1 and Δ_2, hence Δ_1 and Δ_2 are both simply connected.

Let $f_1 : U_1 \to \Delta_1$ be the Riemann map, and let $\Gamma = f_1 H f_1^{-1}$ be the Fuchsian equivalent of H. Define $f_2 : U_2 \to \Delta_2$ by $f_2 = J \circ f_1 \circ j$. By Theorem B, f_1 and f_2 are restrictions of a fractional linear transformation f. Then $J = f \circ j \circ f^{-1}$.

PROOF OF THEOREM C. If $f_1(U_1) \cap f_2(U_2) = \varnothing$, then define

$$\begin{aligned} J(z) &= f_2 \circ j \circ f_1^{-1}(z), && z \in f_1(U_1), \\ &= f_1 \circ j \circ f_2^{-1}(z), && z \in f_2(U_2), \\ &= z, && z \in \Lambda(G). \end{aligned}$$

Note $f_1(U_1)$ and $f_2(U_2)$ are both invariant under G, and so, upon addition of some isolated points, are both simply-connected.

Since $f_i(U_i)$, $i = 1, 2$, is, except for countably many isolated points, a component of G, if $f_1(U_1) \cap f_2(U_2) \neq \varnothing$, then (modulo some isolated points) $f_1(U_1) = f_2(U_2)$. In this case, set

$$\begin{aligned} J(z) &= f_2 \circ j \circ f_1^{-1}(z), && z \in f_1(U_1), \\ &= z, && z \in \Lambda(G). \end{aligned}$$

It is obvious that each of the maps J defined above extend by continuity to the isolated points at which they have not yet been defined.

PROOF OF THEOREM D. By Accola's remark [1], Δ_1 and Δ_2 are both simply-connected. Let $F_1 : \Delta_1 \to U_1$ be the Riemann map, and let $\psi : G \to \Gamma$ be the isomorphism of G onto the Fuchsian group Γ given by $\psi(g) = F_1 \circ g \circ F_1^{-1}$. Using the Fenchel-Nielsen Isomorphism Theorem [5] (see, for example, Marden [8] for a proof) there is a homeomorphism $F_2 : \Delta_2 \to U_2$ with $F_2 \circ g \circ F_2^{-1} = \psi(g)$ for all $g \in G$. By Ahlfors' Finiteness Theorem [2], and Bers' Approximation Theorem [4], F_2 can be chosen to be quasiconformal. Set

$$\begin{aligned} \mu(z) &= \frac{\partial F_2/\partial \bar{z}}{\partial F_2/\partial z}, && z \in \Delta_2, \\ &= 0, && z \notin \Delta_2, \end{aligned}$$

and let w^μ (see Ahlfors-Bers [3]) be a quasiconformal homeomorphism satisfying

$$\partial w^\mu/\partial \bar{z} = \mu \partial w^\mu/\partial z.$$

Then $G^\mu = w^\mu G(w^\mu)^{-1}$ is again a Kleinian group and $w^\mu \circ (F_i)^{-1}$ is conformal in $U_i, i = 1, 2$. Hence $J = w^\mu \circ F_i^{-1} \circ j \circ F_i \circ (w^\mu)^{-1}$ is an anticonformal homeomorphism of $\Omega(G^\mu)$ which commutes with every element of G^μ. By Theorem A, the group G^μ, which is a quasiconformal deformation of G, is Fuchsian.

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