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# MINIMAL REALIZATION OF MACHINES IN CLOSED CATEGORIES

BY J. A. GOGUEN

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There has been recent interest in extending automaton theory to encompass aspects of machine, language, and control theories [**1**], [**2**]. This paper presents a minimal realization theory for discrete-time machines in suitable categories with such applications, but assuming no background outside pure mathematics except perhaps as motivation. Minimal realization is proved in the strong form of an adjunction between behavior and a realization construction generalizing Nerode's [**8**]. We wish to thank Michael Arbib, Lee Carlson, Saunders Mac Lane, and Lotfi Zadeh for their encouragement and/or technical assistance.

If $\boldsymbol{C}$ is a category, $|\boldsymbol{C}|$ denotes its class of objects; and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is written $A \xrightarrow{fg} C$. Functors are written after their arguments.

1. $X$-**automata**. In the first four sections $\boldsymbol{C}$ is a fixed *suitable* category, i.e., closed symmetric monoidal [**7**] with countable coproducts and canoical cofactorizations [**6**]. Let $\boldsymbol{Mon}$ be the category of *monoids* [**7**] in $\boldsymbol{C}$. For $X \in |\boldsymbol{C}|$, let $X^* = \coprod_t \otimes^t X$, the coproduct over the nonnegative integers of iterated "tensor" powers $\otimes^t$ of $X$, where $\otimes$ is the multiplication in $\boldsymbol{C}$. Let $i_0 : I \to X^*$ be the zeroth injection, from $\otimes^0 X = I$, the identity for $\otimes$ in $\boldsymbol{C}$. Finally, define $\mu : X^* \otimes X^* \to X^*$ to be the composite

$$\left(\coprod_r \otimes^r X\right) \otimes \left(\coprod_s \otimes^s X\right) \cong \coprod_{r,s} (\otimes^r X) \otimes (\otimes^s X) \cong \coprod_{r,s} \otimes^{r+s} X \to X^*$$

where the first isomorphism uses the distributivity of $\otimes$ over $\coprod$ which arises from the adjointness of $\otimes$, the second isomorphism is a generalized associative law in $\boldsymbol{C}$, and the third morphism is defined by letting its $\langle r, s\rangle$-component be the $r + s$ injection $\otimes^{r+s} X \to X^*$. Then [**7**], $\langle X^*, \mu, i_0\rangle \in |\boldsymbol{Mon}|$.

For $M \in |\boldsymbol{Mon}|$, let $\boldsymbol{Act}^M$ be the category of right $M$-actions in $\boldsymbol{C}$, that is, $\alpha : S \otimes M \to S$ satisfying appropriate identities [**7**]. For $X \in |\boldsymbol{C}|$, an $X$-*monadic algebra* in $\boldsymbol{C}$ is $\delta : S \otimes X \to S$; and a morphism $h : \delta \to \delta'$ of such algebras is $h : S \to S'$ such that $(h \otimes X)\delta' = \delta h$. Let $\boldsymbol{Mond}^X$ be the resulting category. $\delta$ is often called a *transition* morphism.

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THEOREM 1. $\boldsymbol{Act}^{X^*} \cong \boldsymbol{Mond}^X$.

A *point* for an $M$-action or $X$-monadic algebra is $\sigma: I \to S$ in $\boldsymbol{C}$. Denote the categories of pointed actions and algebras, with morphisms preserving the points, by $\boldsymbol{Act}_\bullet^M$ and $\boldsymbol{Aut}^X$ respectively, the latter for $X$-*automata*. These generalize Kalman's modules in [**5**].

COROLLARY 2. $\boldsymbol{Act}_\bullet^{X^*} \cong \boldsymbol{Aut}^X$.

Let $\langle X^*, \hat{\mu}, i_0 \rangle$ with $\hat{\mu}: X^* \otimes X \to X^*$ be the $X$-automaton corresponding by this corollary to the action $\mu: X^* \otimes X^* \to X^*$ of $X^*$ on itself.

THEOREM 3. *$\langle X^*, \hat{\mu}, i_0 \rangle$ is an initial $X$-automaton.*

If $\langle S, \delta, \sigma \rangle$ is an $X$-automaton, let $\delta^+: X^* \to S$ denote the unique $X$-automaton morphism. Call $\langle S, \delta, \sigma \rangle$ *reachable* iff $\delta^+$ is epic in $\boldsymbol{C}$.

THEOREM 4. *$\boldsymbol{Aut}^X$ has co-images.*

This preliminary material works for somewhat less than suitable categories, since the internal hom is not used. Only Theorem 4 uses canonical cofactorization.

2. **Machines.** A *machine* $M$ in $\boldsymbol{C}$ is $\langle X, S, Y, \delta, \lambda, \sigma \rangle$ such that $\langle S, \delta, \sigma \rangle$ is an $X$-automaton and $\lambda: S \to Y$ is a morphism in $\boldsymbol{C}$. We also fix $X$ and speak of the $X$-machine $\langle S, Y, \delta, \lambda, \sigma \rangle$. $M$ is *reachable* iff its $X$-automaton is. A *morphism* of $X$-machines is a pair $\langle a, b \rangle$ with $a$: $S \to S'$, $b$: $Y \to Y'$ such that $a$ is an $X$-automaton morphism and $\lambda b = a\lambda'$. Denote the category of reachable $X$-machines and morphisms in $\boldsymbol{C}$ by $\boldsymbol{M}^X$. $S$, $Y$ (resp. $a, b$) are the *state* and *output objects* (resp. *components*) of $M$ (resp. $\langle a, b \rangle$). $X$ is the *input* object.

An $X$-*behavior* in $\boldsymbol{C}$ is a morphism $f: X^* \to Y$, and a morphism $f \to f'$ of $X$-behaviors is $b: Y \to Y'$ such that $fb = f'$. Let $\boldsymbol{B}^X$ denote the category. Given an $X$-machine $M$, let $ME = \delta^+\lambda: X^* \to Y$, where $\delta^+$ arises from the $X$-automaton of $M$; given $\langle a, b \rangle: M \to M'$ in $\boldsymbol{M}^X$, let $\langle a, b \rangle E = b$. Then $E: \boldsymbol{M}^X \to \boldsymbol{B}^X$ is a functor, called the *external behavior* functor. Say that $M$ *realizes* $f$ iff $ME = f$.

Given $f: X^* \to Y$, let $fA$ be the $X$-automaton $\langle [X^*, Y], \alpha_f, \sigma_f \rangle$, where $\alpha_f: [X^*, Y] \otimes X \to [X^*, Y]$ and $\sigma_f: I \to [X^*, Y]$ are the adjoint transforms of the composites

$$([X^*, Y] \otimes X) \otimes X^* \cong [X^*, Y] \otimes (X^* \otimes X) \xrightarrow{[X^*, Y] \otimes ((X^* \otimes i_1)\mu)}$$

$$[X^*, Y] \otimes X^* \xrightarrow{\nu} Y,$$

and $I \otimes X^* \cong X^* \xrightarrow{f} Y$ where $i_1: X \to X^*$ is the coproduct injection and $v$ is the "evaluation" morphism or co-unit arising from the adjunction $\otimes \dashv [\ ,\ ]$. Given $b: f \to f'$ in $\boldsymbol{B}^X$, let $bA = [X^*, b]$. Then $A: \boldsymbol{B}^X \to \boldsymbol{Aut}^X$ is a functor. Given $f \in |\boldsymbol{B}^X|$, let $\bar{f}: X^* \to [X^*, Y]$ be the unique $X$-automaton morphism $\langle X^*, \hat{\mu}, i_0 \rangle \to fA$. Let $e_0$ denote the composite $[X^*, Y] \cong [X^*, Y] \otimes I \xrightarrow{[X^*, Y] \otimes i_0} [X^*, Y] \otimes X^* \xrightarrow{v} Y$.

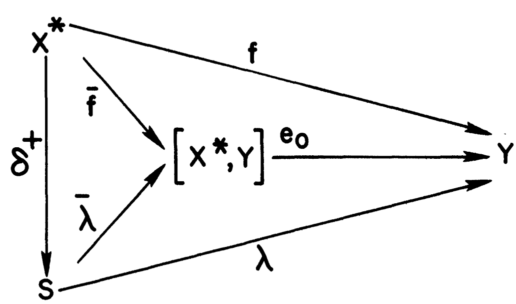
Lemma 5. $\bar{f}e_0 = f$.

Now let $M$ be an $X$-machine, and define $\bar{\lambda}: S \to [X^*, Y]$ to be the adjoint transform of the composite $S \otimes X^* \xrightarrow{\delta^+} S \xrightarrow{\lambda} Y$.

Lemma 6. *$\bar{\lambda}: S \to fA$ is an $X$-automaton morphism, and $\bar{\lambda}e_0 = \lambda$.*

Note that by the uniqueness of $X$-automaton morphisms from $\langle X^*, \hat{\mu}, i_0 \rangle$, we must have $\delta^+ \bar{\lambda} = \overline{ME}$. These results are summarized as follows.

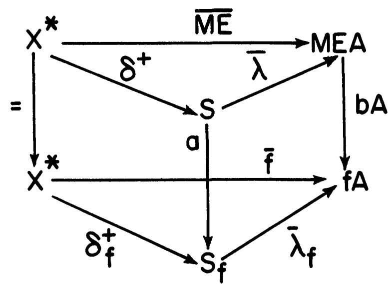
Theorem 7 (Intrinsic Factorization). *Given an X-machine M with behavior $f = ME$, the following commutes in $\boldsymbol{C}$:*



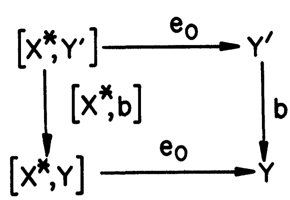

3. **Minimal realization.** Given a behavior $f: X^* \to Y$, let $\langle S_f, \delta_f, \sigma_f \rangle$ be the $X$-automaton coimage of the automaton morphism $\bar{f}: X^* \to [X^*, Y]$ (using Theorem 4). The automaton morphism $X^* \to S_f$ is epic in $\boldsymbol{C}$, and must be $\delta_f^+$ by Theorem 3, so that this $X$-automaton is reachable. Let $\lambda_f$ be the composite $S_f \to [X^*, Y] \to^{e_0} Y$, where the first map is the other part of the co-image factorization of $\bar{f}$ (and incidentally is a strong monic in $\boldsymbol{C}$). Then $fN = \langle S_f, Y, \delta_f, \lambda_f, \sigma_f \rangle$ is the *Nerode* [8] *realization* of $f$. That $fNE = f$ follows from Lemma 5. That $N$ is a functor $\boldsymbol{B}^X \to \boldsymbol{M}^X$ follows automatically from our main result, the universal property for $fN$. We sketch a proof, much of the work being contained in the previous results.

Theorem 8. *Given a suitable category $\boldsymbol{C}$, $X \in |\boldsymbol{C}|$, and an X-behavior f, each morphism $ME \to f$ in $\boldsymbol{B}^X$ with $M \in |\boldsymbol{M}^X|$ is $gE$ for a unique $g: M \to fN$ in $\boldsymbol{M}^X$. Moreover, the state component of g is epic in $\boldsymbol{C}$.*

PROOF. Let $b: ME \to f$ in $B^X$, and consider in $Aut^X$ the diagram




There is a unique $a: S \to S_f$ such that everything commutes, because $\delta^+$ is epic, $\delta^+\bar{\lambda}(bA) = \bar{f}$, and $\delta_f^+$ is the co-image of $\bar{f}$. Because $a$ is an automaton morphism $\langle a, b\rangle : M \to fN$ is in $M^X$ if $\lambda b = a\lambda_f$; but this follows from Lemma 6 by composing the right front square above with




the above square which commutes by naturality of the constructions involved. Now $\langle a, b\rangle E = b: ME \to fNE = f$ as required, and it remains to show $\langle a, b\rangle$ is the only $X$-automaton morphism such that $gE = b$, which follows since $\delta^+$ is epic. □

General lore [7] about adjoints now gives a number of results.

THEOREM 9. *The functor* $E: M^X \to B^X$ *has the Nerode functor* $N: B^X \to M^X$ *as a right adjoint left inverse.*

We say a machine $M$ is *minimal* iff it is reachable and satisfies the following condition for some $f \in |B^X|$: For any reachable $X$-machine $M'$ and any morphism $h: M'E \to f$ in $B^X$, there is a unique $g: M' \to M$ in $M^X$ such that $gE = h$. Then any two minimal realizations of $f$ are isomorphic. We also say $M$ is *reduced* iff the morphism $\bar{\lambda}: X \to [X^*, Y]$ is strong monic [6] in $C$.

THEOREM 10. $fN$ *is a minimal realization of* $f \in |B^X|$.

THEOREM 11. *The full subcategory* $R^X$ *of* $M^X$ *with reduced machines as objects is reflective.*

THEOREM 12. *A machine is minimal iff reduced and reachable.*

Thus, $M$ is a minimal realization of $f$ iff $M$ is reduced, reachable, and realizes $f$. This suggests the following globalization: *A minimal realization functor* is a right adjoint left inverse to the behavior functor. The Nerode function is one by Theorem 9, the proof of which also shows that $N$ is a minimal realization functor iff each $fN$ is reduced (or equivalently by Theorem 12, minimal, since it is already reachable) and realizes $f$. Moreover, since it is a right adjoint, a minimal realization functor preserves all (inverse) limits, for example products, and also monics. The adjunction $E \dashv N$ has a *unit* natural transformation $\eta : \boldsymbol{M}^X \to EN$, called the *reduction transform* since $\eta_M : M \to MEN$ takes a machine to its reduced form; and $N : \boldsymbol{B}^X \to \boldsymbol{M}^X$ is a minimal realization functor iff $NE = \boldsymbol{B}^X$ and there is a natural transformation $\eta : \boldsymbol{M}^X \to EN$ such that $\eta E = E$ and $N\eta = N$. Moreover, in this case each $\eta_M$ has epic state component.

4. **Realization in subcategories.** A pair $\boldsymbol{M}_1, \boldsymbol{B}_1$ of subcategories of $\boldsymbol{M}^X$, $\boldsymbol{B}^X$ is *appropriate* iff the restrictions of $N$ and $E$ to $\boldsymbol{B}_1$ and $\boldsymbol{M}_1$ factor through $\boldsymbol{M}_1$ and $\boldsymbol{B}_1$ (respectively) giving $N_1 : \boldsymbol{B}_1 \to \boldsymbol{M}_1$ right adjoint left inverse to $E_1 : \boldsymbol{B}_1 \to \boldsymbol{M}_1$. Since $E_1$ is a right inverse, it is surjective, so that $\boldsymbol{B}_1 = \boldsymbol{M}_1 E$. Clearly all results of §3 hold for an appropriate pair of subcategories: $fN_1$ is minimal and reduced; $N_1$ preserves limits and monics; the adjunction's unit $\eta^1$ satisfies $\eta^1 E_1 = E_1$ and $N_1 \eta^1 = N_1$; and each $\eta^1_M$ has epic state component.

PROPOSITION 13. *Let $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ be subcategories of $\boldsymbol{C}$, let $X \in |\boldsymbol{C}|$, and let $\boldsymbol{M}_1 = \boldsymbol{M}^X(\boldsymbol{C}_1, \boldsymbol{C}_2)$ be the subcategory of $\boldsymbol{M}^X$ with state components (of machines and morphisms) in $\boldsymbol{C}_1$ and output components in $\boldsymbol{C}_2$. If $\boldsymbol{C}_1$ is closed under quotients (i.e., all epics if $\boldsymbol{C}$ with domain in $\boldsymbol{C}_1$ lie in $\boldsymbol{C}_1$), then $\boldsymbol{M}^X_1$, $\boldsymbol{B}^X_1 = \boldsymbol{M}^X_1 E$ is an appropriate pair.*

For $X, Y \in |\boldsymbol{C}|$ and $\boldsymbol{C}_1$ a subcategory of $\boldsymbol{C}$, let $\boldsymbol{M}^{X,Y}(\boldsymbol{C}_1) = \boldsymbol{M}^X(\boldsymbol{C}_1, \{Y\})$, where $\{Y\}$ is the subcategory of $\boldsymbol{C}$ with only the object $Y$ and its identity. Write $\boldsymbol{M}^{X,Y}$ for $\boldsymbol{M}^{X,Y}(\boldsymbol{C})$.

COROLLARY 14. *$\boldsymbol{M}^{X,Y}, \boldsymbol{M}^{X,Y}E$ is an appropriate pair; and if $\boldsymbol{C}_1$ is closed under quotients, so is $\boldsymbol{M}^{X,Y}(\boldsymbol{C}_1)$, $\boldsymbol{M}^{X,Y}(\boldsymbol{C}_1)E$.*

5. **Applications.** The category of sets is suitable with binary Cartesian product, giving a minimal realization theory for discrete (possibly infinite) machines. The full subcategory of finite sets is closed under quotients, so Proposition 13 gives the theory for standard finite automata. In this case, a realization $M$ of $f$ is minimal in our sense iff its state set has minimal cardinality among all finite realizations of $f$. Most results in §3

are new even for this well-studied case; see [4] for further discussion.

If $Y$ is the two point set $\{0, 1\}$ and $X$ is finite, then Corollary 14 gives an adjunction between finite state (or regular) languages and minimal acceptors.

The category of modules over a commutative ring $K$ with unit is suitable with tensor product $\otimes_K$, and its full subcategory of finitely generated modules is closed under quotients. Because $\delta: S \otimes_K X \to S$ linear corresponds to $\delta: S \times X \to S$ bilinear, we call the machines in this theory *bilinear transition* machines. Actually, it is better for concrete applications to let $\boldsymbol{C}$ be the full subcategory with all $K^S$ as objects, where $S$ is a set giving a distinguished basis for the free module $K^S$. This is also suitable. The special case where $K = Y = \mathbf{2}$, the two point field, gives an adjunction between bilinear transition acceptors and (arbitrary) languages having the basis for $X$ as alphabet, since these correspond naturally to linear $f: X^* \to \mathbf{2}$.

An *affine* morphism $A \to B$ between modules is a function of the form $f + b$, where $f: A \to B$ is linear and $b \in B$. The category $\boldsymbol{Aff}_K$ of modules over $K$ with affine morphisms is suitable with *affine tensor product* $A \circledA_K B = A \otimes_K B + A + B$. We call the machines and behaviors: affine since $\delta: S \circledA_K X \to S$ affine corresponds to $\delta: S \times X \to S$ biaffine. The subcategory of finitely generated modules is closed under quotients. Again, it is better for applications to use the suitable subcategory with objects $K^S$. Indeed, these machines seem to give a better and richer model for physical systems than the usual linear systems. Because any linear $\delta: X \times X \to S$ is biaffine, the linear machines form a subcategory. But a minimal affine realization of a linear behavior can be simpler than any linear realization. Moreover, affine machines are richer in (loosely speaking) allowing multiplications between inputs and states. This type of machine and its minimal realization theory seem to be new.

The category ***Kell*** of all Kelley (often called compactly generated $T_2$) spaces [3] with binary Cartesian product is suitable. This gives a new minimal realization theory for continuous (generally nonlinear) machines. Here minimality cannot be expressed the traditional way, that some numerical characteristic of the state space is least, because Kelley spaces admit no such convenient number.

Finally we remark that the restriction to a fixed input object $X$ can be removed; see [4].

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Committee on Information Sciences, University of Chicago, Chicago, Illinois 60637

Mathematical Science Department, Thomas J. Watson Research Center, I.B.M., Yorktown Heights, New York 10598 (current address)