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STUDY OF THE PERMANENT CONJECTURE AND SOME GENERALIZATIONS

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Let K be a convex polyhedron in an affine space with a set of extreme points $\mathscr{E}$. A set $\mathscr{S} = \{A_1, A_2, \ldots, A_n\}$ of affine functions, none of them identically zero on K, is said to determine K if there exists an affine subspace H such that $K = \{x \in H \mid A_i(x) \geqq 0 \forall_i\}$. Let $\boldsymbol{R}_0^n$ be closed first 2^n-gant in $\boldsymbol{R}^n$. $\boldsymbol{R}_0^n$ is a multiplicative semigroup and for two points u and v in $\boldsymbol{R}_0^n$ we define additionally $u^v = u_1^{v_1} \cdot u_2^{v_2} \ldots u_n^{v_n}$. $0^0 = 1$. If α is a scalar $\geqq 0$, we define $u^\alpha = (u_1^\alpha, u_2^\alpha, \ldots, u_n^\alpha)$. Also we define a map $A: K \to \boldsymbol{R}_0^n$ by $A(x) = (A_1(x), A_2(x), \ldots, A_n(x))$. Let c be a strictly positive function on $\mathscr{E}$. For $y \in \boldsymbol{R}_0^n$, define

$$Q(y) = \sum_{e \in \mathscr{E}} c(e) y^{A(e)}$$

and, for $x \in K$, define $P(x) = Q(A(x))$. P is strictly positive, so it is of some interest to find its minimum.

If K is the set D_k of $k \times k, k \geqq 2$, doubly stochastic matrices, and we take for $\mathscr{S}$ the coordinate functions and take $c \equiv 1, P(x)$ is the permanent of x, $\operatorname{Perm}(x) = \sum x^\pi$, where the summation is over the permutation matrices.

Returning now to the general case, define a map $q: \boldsymbol{R}_0^n \to K$ by

$$q(y) = \frac{1}{Q(y)} \sum_{e \in \mathscr{E}} c(e) y^{A(e)} e,$$

defined for $Q(y) \neq 0$, and a map $h: K \to K$ by $h(x) = q(A(x))$. Then

THEOREM 1. *h is a bijection.*

Q is homogeneous of degree d if $\sum_i A_i(e) = d$ for all $e \in \mathscr{E}$. Since the sum of the inner normals to the faces of K, with lengths equal to the area of the faces, is zero, homogeneity can always be achieved by appropriate choice of $\mathscr{S}$. In case Q is homogeneous, we have additionally

THEOREM 2. *$P(h(x)) \geqq P(x)$, with equality only for $h(x) = x$.*

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The permanent is homogeneous of degree k, so the map h above is of considerable interest. For one thing, we have the amusing statement that every $y \in D_k$ may be written in the form

$$y = \sum_{\pi} x^{\pi}\pi \Big/ \sum_{\pi} x^{\pi}$$

for a unique choice of doubly stochastic x. It would be interesting to obtain an intrinsic characterization of the representation of y above in the class of all representations of y as a convex sum of permutation matrices. For another, the inverse to the map h does not increase the permanent, but except for the case $k = 2$, it appears difficult to get useful expressions for the inverse. There is another description of the map h which is also useful. Let $x \in D_k$, and let X_{ij} be the $k - 1$ by $k - 1$ minor associated to x_{ij}. Then

$$h_{ij}(x) = \frac{x_{ij}\,\mathrm{Perm}(X_{ij})}{\mathrm{Perm}(x)}.$$

In the study of the homogeneous case, one is led quite naturally to the following considerations. Let L be the set of $l \in \boldsymbol{R}_0^n$ such that $l^{A(e)} \geqq 1$ $\forall e \in \mathscr{E}$, which is the same as saying $l^{A(x)} \geqq 1$ $\forall x \in K$. L is a multiplicative semigroup, but also a convex set, owing to concavity of the logarithm. Define $L_1 \subset L$ as the set of l such that $l^{L(e)} = 1$ $\forall e \in \mathscr{E}$. L_1 is then a group, and it may be proved that L_1 is the set of extreme points of L, but L is not necessarily the convex hull of L_1. For $y \in \boldsymbol{R}_0^n$ define

$$E(y) = \frac{1}{d} \min_{l \in L} \sum_i y_i l_i.$$

The minimum is always achieved as a matter of fact for a point of L_1, or as the limit along a sequence of points of L_1.

THEOREM 3. (i) *If* $Q(y) = 0$, $E(y) = 0$, *otherwise*

$$E(y) = [Q(y)/P(h^{-1}(q(y)))]^{1/d}.$$

(ii) *For* $u, v \in \boldsymbol{R}_0^n$, $E(u + v) \geqq E(u) + E(v)$ *and* $E(u^p v^q) \leqq E^p(u)E^q(v)$, $p + q = 1, p, q \geqq 0$.

The last two properties follow from the definition and the semigroup property of L. From (i) we see that $E(y)$ is continuous and that $E(A(x)) \equiv 1$ for $x \in K$.

It is of interest to investigate the cases when $E(y)$ is an attained minimum. To this end we say y has K-like support if there exists $x \in K$ such that $y_i = 0$ if and only if $A_i(x) = 0$. We say $x \in K$ is indecomposable if, for each i, $\exists e \in \mathscr{E}$ for which $A_i(e) \neq 0$ and such that $\prod_{v \neq i} A_v(x)^{A_v(e)} \neq 0$. y with K-like support is called indecomposable if the associated $x \in K$ is indecomposable.

THEOREM 4. *If y has K-like support, $\sum y_i l_i$ has an attained minimum, and l_i is uniquely determined if $y_i \neq 0$. (With minor qualifications, the converse of the last statement is also true.) If y is indecomposable, the minimum is uniquely attained.*

By examining the properties of the attained minimum, we have

THEOREM 5. *If y has K-like support, there exists a positive constant α, an $l \in L_1$, and unique $x \in K$ such that $y = \alpha A(x) l$, and $\alpha = E(y)$. If y is indecomposable, l is also unique.*

From this there follows

THEOREM 6.

$$E(y) = \max_{x \in K} [y^{A(x)}/A(x)^{A(x)}]^{1/d}.$$

Also,

THEOREM 7. *If M and m are the upper and lower bounds for $P(x)$, then $ME^d(y) \geqq Q(y) \geqq mE^d(y)$.*

For D_k, the elements of L_1 are just matrices whose i,j entry is $\lambda_i u_j$ with $\prod_i \lambda_i = \prod_i u_i = 1$. And x is indecomposable if it may not be written as a reduced matrix after some permutation of the rows and another of the columns. The above reduces to the known theorem that if y has the same support as a doubly stochastic matrix, there exist positive diagonal matrices D_1 and D_2 such that $D_1 \times y \times D_2$ is doubly stochastic, where the multiplication in last is matrix multiplication.

If $Q(y) = \sum c(e) y^{A(e)}$ is homogeneous of degree $d \leqq 1$, then it is easy to see that $Q(y)$ is a concave function, since each of the summands is, so the minimum of $P(x)$ is attained at an extreme point. If $d \geqq 1$, define

$$Q_1(y) = \sum c(e) y^{A(e)/d}$$

so that we have $Q_1(A(x)) \geqq \min_{e \in \mathscr{E}} c(e) A(e)^{A(e)/d} = \lambda$.
Now, for $x \in K$, write $A(x)^{1/d} = \alpha^1 A(y)$ with $y \in K$, $l \in L_1$, $\alpha > 0$, and then $\alpha = E(A(x)^{1/d}) = 1/E^{1/d}(A^d(y))$. Every $y \in K$ occurs for some $x \in K$. Hence

$$Q_1(A(x)) = \sum c(e) \alpha^d A(y)^{A(e)} = \alpha^d P(y),$$

so $P(y) \geqq \lambda E(A^d(y))$. Let I be the point in R_0^n with all coordinates equal 1. Then

$$1 = E(A(y)) = E(A(y) \cdot I) \leqq E^{1/d}(A^d(y)) E^{(d-1)/d}(I),$$

so

$$E(A^d(y)) \geqq \frac{1}{E^{d-1}(I)} = \min_{x \in K} [A(x)^{A(x)}]^{(d-1)/d} \geqq \left(\frac{d}{n}\right)^{d-1}$$

THEOREM 8. $P(x) \geqq \lambda(\frac{d}{n})^{d-1}, x \in K$.

THEOREM 9. $\mathrm{Perm}(x) \geqq 1/k^{k-1}; x \in D_k$.

This is a far cry from the van der Waerden conjecture, but better, I believe, than other available results.

By purely combinatorial arguments we can obtain a result in some respects better than the last. Let β = integral part of $k^{(k-1)/k}$. Then the sum of the β largest terms in the expansion of $\mathrm{Perm}(x)$, $x \in D_k$, is $\geqq \beta/k^k$, with equality only for the matrix with all equal entries.

If the permanent conjecture is true, then it follows as well that $\mathrm{Perm}(x^r)$, $r \geqq 1$, achieves its minimum at the matrix with all equal entries. Using the mapping function h associated to $\mathrm{Perm}(x^r)$ we can prove

THEOREM 10. *$\exists r$, depending on k, so that* $\mathrm{Perm}(x^r)$ *for $x \in D_k$ achieves its minimum uniquely at the matrix with equal entries.*

Proofs of all the above will appear elsewhere.

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