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ON ALGEBRAIC VARIETIES WHOSE UNIVERSAL COVERING MANIFOLDS ARE COMPLEX AFFINE 3-SPACES

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1. **Introduction.** Let V be a nonsingular projective algebraic variety defined over the field of complex numbers. By $\tilde{V}$ we denote the universal covering manifold of V. It is clear that if $\tilde{V}$ is an abelian variety, then $\tilde{V}$ turns out to be a complex affine space. The author is concerned with a converse of this fact. Thus, he proposes the following:

CONJECTURE U_n. Suppose that $\tilde{V}$ is a complex affine n-space. Then there exists an abelian variety which is a finite unramified covering manifold of V.

This has been solved only for $n = 1, 2$. We note that the proof for $n = 2$ requires a detailed study of the classification of algebraic surfaces. In his thesis [3], the author introduced the notion of Kodaira dimension $\kappa(V)$ of algebraic varieties V and by using it he intends to extend the classification theory into higher dimensional case (see [5]). In this note, he shall give a sketchy proof of the following partial solution of U_3.

THEOREM. *Suppose that V satisfies the hypothesis for U_3. Then $\kappa(V) \neq 1$ and* 3.

The detailed proof and related results will appear elsewhere.

2. **Divisor-dimension and Kodaira dimension.** We recall definitions and some results concerning divisor-dimension and Kodaira dimension (see [3]). Let V be a complete algebraic variety and D a Cartier divisor on V. Denoting by $\mathcal{O}(D)$ the invertible sheaf associated with D, we define $l(D)$ to be $\dim \Gamma(V^*, \mu^*\mathcal{O}(D))$ where $\mu: V^* \to V$ is a normalization of V. We study $l(mD)$ as a function of m for sufficiently large integer m. If there exists a positive integer m_0 such that $l(m_0 D) > 0$, we can find real positive constants α, β and a nonnegative integer κ which satisfy

$$\alpha m^\kappa \leqq l(m \cdot m_0 D) \leqq \beta m^\kappa$$

for sufficiently large values of m. Since the κ is independent of the choice

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of α, β, and m_0, we define D-dimension of V, written $\kappa(D, V)$, to be the κ. If $l(mD) = 0$ for any $m \geqq 1$, we set $\kappa(D, V) = -\infty$. Clearly, we have $\kappa(D, V) = \kappa(rD, V)$ for any $r \geqq 1$. Hence, we can define $\kappa(D, V)$ for a fractional divisor D. Now let V be an algebraic variety. By Hironaka, there exists a nonsingular projective model V^* of V. We indicate by $K(V^*)$ a canonical divisor of V^*. Define the m-genus $P_m(V)$ to be $l(mK(V^*))$ and also define the Kodaira dimension $\kappa(V)$ to be $\kappa(K(V^*), V^*)$. Suppose that $\kappa(V) \geqq 0$. Then we can find a fiber space $f: V^* \to W$, V^*, W being non-singular projective algebraic varieties with the following properties:

(i) V^* is birationally equivalent to V,
(ii) $\dim W = \kappa(V)$,
(iii) any general fiber V_w^* is irreducible;
(iv) $\kappa(V_w^*) = 0$.

Moreover, these properties characterize $f: V^* \to W$ up to birational equivalence. Hence, we call $f: V^* \to W$ a canonical fiber space associated to V.

3. **A proof of** U_2. First, we notice some basic properties which V has, if V satisfies the hypothesis for U_n.

PROPOSITION 1. *Let W be a subvariety of V and W^* a nonsingular model of W. Then the fundamental group of W^* is infinite.*

Hence, there are no rational curves on V. This implies that V is strongly minimal (for the definition, see [**9**]).

PROPOSITION 2 (KODAIRA). $\kappa(V) < n$.

For the proof, we refer to [**6**].

By using these, we shall sketch the proof of U_2.

Case I. $\kappa(V) = 1$. In this case, we shall derive a contradiction in the following five steps. (α) By a theorem due to the Italian school, we see the existence of an elliptic fiber space $f: V \to W$. That is to say, V is an elliptic surface. (β) Any singular fiber of an elliptic surface consists of rational curves or is a multiple of an elliptic curve (see the table of singular fibers in [**7**]). Hence, the singular fibers $f^*(a_1), \ldots, f^*(a_s)$ of the elliptic surface V are multiples of elliptic curves $f^{-1}(a_1), \ldots, f^{-1}(a_s)$, respectively. Thus, we have $f^*(a_1) = e_1 f^{-1}(a_1), \ldots, f^*(a_s) = e_s f^{-1}(a_s)$. ($\gamma$) The canonical bundle formula (see [**8**], [**2**]) reads

$$k(V) = \kappa(K(W) + \sum (1 - 1/e_j) a_j, W).$$

Therefore, $2\pi - 2 + \sum (1 - 1/e_j) > 0$ follows, where π is the genus of W. (δ) We can construct the universal covering manifold W^* which ramifies at every point over each a_j with the multiplicity e_j for any $1 \leqq j \leqq s$. Then, $V_1 = V \times_W \tilde{W}^*$ is an unramified covering manifold of V. As a consequence, (ε) we obtain a surjective holomorphic mapping from $C^2 = \tilde{V}_1$ onto $\tilde{W}^*$, a complex upper half plane. On the other hand, in view of the Liouville theorem, we see that f is constant.

Case II. $\kappa(V) = 0$. By the classification theory of algebraic surfaces, if $\pi_1(V)$ is infinite, then V is an abelian variety or a hyperelliptic surface which has an abelian variety as a finite unramified covering manifold.

Case III. $\kappa(V) = -\infty$. In this case, from the Enriques criterion we deduce immediately that V is a ruled surface. Therefore, V has many rational curves. This contradicts Proposition 1.

4. **The existence of minimal canonical fiber space.** In this section, we shall state some analogues for the steps (α) and (β) in Case I.

PROPOSITION 3. *Let V be a minimal algebraic variety of dimension* 3. *Suppose that $\kappa(V) = 1$. Then there exists a canonical fiber space $f: V \to W$ whose general fiber is a minimal surface.*

PROPOSITION 4. *Under the same assumption as above, we further assume that V is strongly minimal. Then every singular fiber of the fiber space $f: V^* \to W$ has only one irreducible component. A singularity of the irreducible component is negligible. Moreover, if a general fiber is an abelian variety or a hyperelliptic surface, singular fibers are multiples of nonsingular surfaces.*

PROPOSITION 5. *Let V be a strongly minimal algebraic variety of dimension* 3. *Suppose that $\kappa(V) = 2$. Then there exists a canonical fiber space $f: V \to W$ such that W is relatively minimal and such that every fiber is (possibly a multiple of) an elliptic curve.*

We call the fiber space, constructed in Proposition 3, the minimal canonical fiber space associated with V.

In the proofs of these propositions, the following lemmas are useful.

LEMMA 1. *Let $f: V \to W$ be a fiber space of nonsingular projective algebraic varieties such that* $\dim W = 1$ *and* $\kappa(V) \geqq 0$. *Suppose that a pluri-canonical divisor of a general fiber is linearly equivalent to zero. Then some irreducible component C_ν of any reducible fiber $f^*(a) = \sum_{\nu=1}^{s} n_\nu C_\nu$, $s \geqq 2$, has the Kodaira dimension $-\infty$.*

LEMMA 2. *Let $f: V \to W$ be an elliptic fiber space of nonsingular projective algebraic varieties such that* $\dim V = 3$ *and* $\dim W = 2$. *Then a surface contained in a fiber is a rational surface.*

5. The canonical bundle formulas and the proof of $k(V) \neq 1$. Let V satisfy the hypothesis for U_3. Besides, we assume $\kappa(V) = 1$. Then the minimal canonical fiber space has an abelian variety or a hyperelliptic surface as its general fiber. Hence we can easily prove the canonical bundle formula:

$$\kappa(V) = \kappa\Big(K(W) + \sum(1 - 1/e_j)a_j, W\Big).$$

Thus by the assumption we obtain $2\pi - 2 + \sum(1 - 1/e_j) > 0$. Here the notation is the same as in step (γ) in Case I. Following the argument in the steps (δ) and (ε), we can easily derive a contradiction.

In the case when $\kappa(V) = 2$, there is a canonical fiber space $f: V^* \to W$ which has the following property: There exist nonsingular curves $\Delta_1, \ldots, \Delta_s$ on W which satisfy:

(1) for any $w \in W - \bigcup \Delta_i$, the fiber $f^*(w)$ is regular and for any $a_i \in \Delta_i$, the fiber $f^*(a_i)$ is an e_i-tuple of an elliptic curve;

(2) $\Delta_i \cap \Delta_j = \varnothing$ for any $i \neq j$.

The following canonical bundle formula is established:

$$\kappa(V) = \kappa\Big(K(W) + \sum(1 - 1/e_j)\Delta_j, W\Big).$$

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