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NONNOETHERIAN COMPLETE INTERSECTIONS

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Let all rings here be commutative and unitary. As it is well known, a noetherian local ring A (quotient of a regular ring with residue field K) is a complete intersection if and only if the integers

$$\beta_i = \dim_K \mathrm{Tor}_i^A(K, K)$$

appear in an equality of formal series of the following type

$$\sum \beta_i x^i = (1 + x)^r/(1 - x^2)^s.$$

Furthermore the integer $r - s$ is positive (equal to the dimension of the noetherian ring A). In the nonnoetherian case, by means of André-Quillen homology theory, a criterion is given for characterizing the local rings for which the integers β_i are defined and appear in an equality of formal series as above. An example shows that there is no relation between the integers r and s in the nonnoetherian case.

1. **Result.** Let us consider a local ring A with residue field K and its homological invariants $\mathrm{Tor}_i^A(K, K)$ and $H_j(A, K, K)$ (see [1] for the definition). They are related by the following result, among others.

THEOREM. *All the dimensions β_i of the vector spaces* $\mathrm{Tor}_i^A(K, K)$ *are finite and satisfy an equality*

$$\sum \beta_i x^i = (1 + x)^r/(1 - x^2)^s$$

if and only if all the dimensions δ_j of the vector spaces $H_j(A, K, K)$ are finite and satisfy an equality

$$\sum \delta_j x^j = rx + sx^2.$$

PROOF. The proof is given elsewhere in the paper and involves simplicial theory.

REMARK. In the case of characteristic 0, the theorem is a corollary of a result proved by D. Quillen: The graded vector space $H_*(A, K, K)$ is isomorphic to the graded vector space of the indecomposable elements of the graded Hopf algebra $\mathrm{Tor}_*^A(K, K)$. In the case of characteristic p, such a result cannot hold in all degrees for all rings even if divided powers are considered in the definition of the graded vector space of the indecomposable elements.

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REMARK. Actually the theorem is a special case of the following result. All the vector spaces $H_j(A, K, K)$, with $j \geqq 3$, are equal to 0 if and only if the graded algebra with divided powers $\mathrm{Tor}_*^A(K, K)$ is free with generators in degrees 1 and 2 only.

PROPOSITION. *There exists a local ring fulfilling the conditions of the theorem for any pair of integers $r \geqq 0$ and $s \geqq 0$.*

PROOF. Let us consider the tensor product T of r copies of the K-algebra R (see the remark below) and of s copies of the K-algebra S (see the lemma below). We obtain (see [**1**, Proposition 19.3]) an augmented K-algebra with the following homology: The dimension of the vector space $H_j(K, T, K)$ is equal to r for $j = 0$, to s for $j = 1$ and to 0 for $j \geqq 2$. Or equivalently (see [**1**, Proposition 18.2]) the dimension of the vector space $H_j(T, K, K)$ is equal to 0 for $j = 0$, to r for $j = 1$, to s for $j = 2$ and to 0 for $j \geqq 3$. Finally (see [**1**, Proposition 20.2]), the proposition is proved by the isomorphism

$$H_*(T, K, K) \cong H_*(T_I, K, K)$$

where I is the augmentation ideal.

REMARK. There exists an augmented K-algebra R for which the dimension of the vector space $H_j(K, R, K)$ is equal to 0 for $j \neq 0$ and to 1 for $j = 0$. It suffices (see [**1**, Corollary 16.3]) to consider a free K-algebra with one generator.

LEMMA. *There exists an augmented K-algebra S for which the dimension of the vector space $H_j(K, S, K)$ is equal to 0 for $j \neq 1$ and to 1 for $j = 1$.*

PROOF. Let us consider the K-algebra X with the following generators and relations:

$$t^\gamma \quad (0 < \gamma \text{ rational}) \quad \text{and} \quad t^\alpha t^\beta = t^{\alpha+\beta}$$

and with the augmentation mapping t^γ onto 0. This K-algebra X is the union of free K-algebras. Consequently (see [**1**, Proposition 16.2 and Corollary 16.3]), we get the equality

$$H_j(K, X, K) = 0, \qquad j \geqq 0,$$

even for $j = 0$, since all K-derivations of X into K are equal to 0. Now let us consider the ring S equal to X/t^1X. Since the element t^1 does not divide 0 in the ring X, we obtain the following isomorphisms:

$$\begin{aligned} H_j(X, S, K) &\cong 0 \quad \text{for } j \neq 1, \quad \text{and} \\ &\cong K \quad \text{for } j = 1. \end{aligned}$$

Consequently the lemma is proved by the exact sequence

$$H_j(K, X, K) \to H_j(K, S, K) \to H_j(X, S, K) \to H_{j-1}(K, X, K)$$

(see [**1**, Proposition 18.2]).

2. **Proof (Part I).** The following results are used in the proof of the theorem. The base field K is fixed.

Dold's result. Let us consider the nth symmetric product functor S_n^K. If a morphism of simplicial vector spaces gives an epimorphism of graded vector spaces

$$H_*[P_*] \to H_*[Q_*]$$

then it gives an epimorphism of graded vector spaces

$$H_*[S_n^K P_*] \to H_*[S_n^K Q_*].$$

Dold-Thom's result. Let us consider the symmetric algebra functor S^K. For a simplicial vector space P_* with the following homology

$$H_i[P_*] \cong 0 \quad \text{if } i \neq n, \text{ and}$$
$$H_n[P_*] \cong K,$$

there is an isomorphism of graded vector spaces

$$H_*[S^K P_*] \cong H_*(K(Z, n), K)$$

(singular homology with coefficients in K of the Eilenberg-Mac Lane space $K(Z, n)$).

Cartan's result. On the one hand, the vector space $H_n(K(Z, 1), K)$ is isomorphic to K for $n \leqq 1$ and to 0 for $n \geqq 2$. On the other hand, the vector space $H_n(K(Z, 2), K)$ is isomorphic to K for n even and to 0 for n odd.

NOTATION. According to [**1**, Chapter 24] and to [**5**, Theorem 6.12], let us choose with a local ring A and its residue field K, a simplicial K-algebra R_* and a simplicial ideal J_* with the following properties:

(a) there is a natural isomorphism of graded vector spaces

$$H_*[R_*] \cong \operatorname{Tor}_*^A(K, K);$$

(b) there is a natural isomorphism of graded vector spaces

$$H_*[J_*/J_*^2] \cong H_*(A, K, K);$$

(c) there is a natural isomorphism of simplicial vector spaces, for $n \geqq 0$,

$$J_*^n/J_*^{n+1} \cong S_n^K(J_*/J_*^2);$$

(d) for $k < n$, the vector space $H_k[J_*^n]$ is equal to 0.

PROOF. Now let us prove that the condition of the theorem is sufficient. For any $k \geqq 0$, let us consider the homomorphism of vector spaces

$$H_k[J_*] \to H_k[J_*/J_*^2].$$

It is always a surjection. For $k = 1$ or 2, this is well known (see [**1**, Proposition 25.1 and Proposition 26.1]) and for $k \geqq 3$, this is the hypothesis

$$H_k[J_*/J_*^2] \cong H_k(A, K, K) \cong 0.$$

Consequently, we get an epimorphism, for any $k \geqq 0$ and for any $n \geqq 0$,

$$H_k[S_n^K J_*] \to H_k[S_n^K J_*/J_*^2].$$

Then the homomorphism

$$H_k[J_*^n] \to H_k[J_*^n/J_*^{n+1}]$$

must be an epimorphism for any $k \geqq 0$ and for any $n \geqq 0$. Now let us consider the exact sequences

$$0 \to H_k[J_*^{n+1}] \to H_k[J_*^n] \to H_k[S_n^K J_*/J_*^2] \to 0.$$

By means of the convergence theorem (see property (d)), these exact sequences give an isomorphism of graded vector spaces

$$H_*[J_*^0] \cong H_*[S^K J_*/J_*^2].$$

On the left side, we get the graded vector space $\mathrm{Tor}_*^A(K, K)$. On the right side, we get a graded vector space with the Poincaré series $(1 + x)^r/(1 - x^2)^s$ where r is the dimension of $H_1(A, K, K)$ and where s is the dimension of $H_2(A, K, K)$.

3. **Proof (Part II).** Once more we have to use Dold's work (and the Eilenberg-Zilber theorem for the case $k = m$).

LEMMA. *If a morphism of simplicial vector spaces gives epimorphisms of vector spaces*

$$H_k[P_*] \to H_k[Q_*], \qquad k = 0, 1, \ldots, m - 1,$$

(*the zero homomorphism in degree* $k = 0$), *then it gives epimorphisms of vector spaces*

$$H_k[S_n^K P_*] \to H_k[S_n^K Q_*], \qquad 0 \leqq k \leqq m, 0 \leqq n,$$

with one exception: $k = m$ *and* $n = 1$.

REMARK. The isomorphism (see [1, Proposition 25.1])

$$\operatorname{Tor}_1^A(K, K) \cong H_1(A, K, K)$$

and the exact sequence (see [1, Proposition 26.1])

$$0 \to \operatorname{Tor}_1^A(K, K) \wedge \operatorname{Tor}_1^A(K, K) \to \operatorname{Tor}_2^A(K, K) \to H_2(A, K, K) \to 0$$

prove that the numbers δ_1 and δ_2 are finite if the numbers β_1 and β_2 are finite, and that the following equalities hold:

$$\beta_1 = \delta_1 \quad \text{and} \quad \beta_2 = \delta_2 + \beta_1(\beta_1 - 1)/2.$$

REMARK. The graded algebra $\operatorname{Tor}_*^A(K, K)$ is always a free algebra with divided powers. If its Poincaré series (when it exists is equal to $(1 + x)^r/(1 - x^2)^s$, then the generators lie in degrees 1 and 2. Consequently the vector space $\operatorname{Tor}_k^A(K, K)$ with $k \geqq 3$ is generated by products and divided powers. Products and divided powers can be defined in a simplicial way. Finally we can prove that the homomorphism of vector spaces

$$H_k[J_*^2] \to H_k[J_*]$$

is an epimorphism for $k \geqq 3$.

PROOF. Now let us prove that the condition of the theorem is necessary. It suffices to prove the following result. If all the dimensions β_i of the vector spaces $\operatorname{Tor}_i^A(K, K)$ are finite and give an equality

$$\sum \beta_i x^i = (1 + x)^r/(1 - x^2)^s$$

and if the following vector spaces are equal to 0,

$$H_k(A, K, K) = 0, \qquad k = 3, 4, \ldots, m - 1,$$

then the vector space $H_m(A, K, K)$ is equal to 0.

Let us use the same argument as at the beginning of the proof in Part I. Let us consider the exact sequences we obtain, for $n \geqq 2$,

$$0 \to H_{m-1}[J_*^{n+1}] \to H_{m-1}[J_*^n] \to H_{m-1}[S_n^K J_*/J_*^2] \to 0.$$

For $n = 1$, we have the exact sequence

$$0 \to \Omega \to H_{m-1}[J_*^2] \to H_{m-1}[J_*] \to 0$$

where Ω is the cokernel of the homomorphism

$$\omega: H_m[J_*] \to H_m[J_*/J_*^2].$$

By hypothesis, the vector spaces $H_{m-1}[J_*^0]$ and $H_{m-1}[S^K J_*/J_*^2]$ have the same dimensions. Consequently Ω is equal to 0. The homomorphism ω must be a zero epimorphism and the vector space

$$H_m[J_*/J_*^2] \cong H_m(A, K, K)$$

must be equal to 0. Rewrite carefully the proof for $m = 3$. The theorem is proved.

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