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THE FUNDAMENTAL FORM OF A FINITE PURELY INSEPARABLE FIELD EXTENSION

BY MURRAY GERSTENHABER[1]

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The purpose of this note is to show that to every finite purely inseparable field extension K/k there is associated in a natural way a symmetric cochain $f:K \times \ldots \times K$ (γ times) $\to K$ of K with coefficients in itself which we call the "fundamental form" of K. Its degree, γ, depends on certain structural properties of K. The fundamental form is a derivation when considered as a function of any one variable, all others being held fixed. (It is almost always a coboundary when viewed as a function of all variables.) If K is a tensor product of two intermediate fields then its fundamental form is a certain symmetric product of the forms of the intermediate fields. A weak converse is known and a strong one conjectured.

References in this note to Nakai are to [**4**] and [**5**], those to Keith are to [**3**].

1. DEFINITION. Let A be a commutative k-algebra, set $Y^1(A) = \operatorname{End}_k A$ and for every $n > 1$ let $Y^n(A) = Y^n$ be the set of those n-cochains f of A with coefficients in itself which are symmetric as functions of all n variables and which have the property that if all but two variables are fixed then f is a two-cocycle when considered as a function of the remaining ones. If $f \in Y^n$, then the $n+1$-cochain Δf defined by $\Delta f(a_1, \ldots, a_{n+1}) = a_n f(a_1, \ldots, a_{n-1}, a_{n+1}) - f(a_1, \ldots, a_{n-1}, a_n a_{n+1}) + a_{n+1} f(a_1, \ldots, a_n)$ is in Y^{n+1}. This defines the "Nakai operator" $\Delta: Y^n \to Y^{n+1}$. It is easy to verify that for odd n, Δ is identical with the Hochschild coboundary operator δ restricted to Y^n. However, in general, $\Delta^2 \neq 0$ and the Y^i do not form a complex. Those elements of Y^1 which are annihilated by Δ^i are called "ith order derivations" or simply "i-derivations" and form an A-module denoted by $\mathscr{D}^i$. A 1-derivation is an ordinary derivation of A into itself. If A is unital, which we henceforth assume, then we denote by Y_0^n the submodule of Y^n consisting of those cochains in Y^n which vanish when any variable equals 1. We then have $\Delta Y_0^n \subset Y_0^{n+1}$, and $\mathscr{D}^i \subset Y_0^1$ for all i. If $\varphi \in \mathscr{D}^i$, $\psi \in \mathscr{D}^j$ then their composite $\varphi\psi$ is in $\mathscr{D}^{i+j}$ (Nakai). The space $\bigcup_{i=1}^{\infty} \mathscr{D}^i$ of all "high order derivations" is thus a ring with an increasing filtration. When A is a finite purely inseparable field extension

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K of k Nakai has shown that $\bigcup \mathscr{D}^i = Y_0^1$, so for some integer γ one has $\mathscr{D}^1 \subset \mathscr{D}^2 \subset \ldots \subset \mathscr{D}^\gamma = \mathscr{D}^{\gamma+1} = \ldots = (\mathrm{End}_k K)_0$. In this case we also have that $\varphi \in \mathscr{D}^i$ if and only if $\delta\varphi \in \mathscr{D}^1 \cup \mathscr{D}^{i-1} + \mathscr{D}^2 \cup \mathscr{D}^{i-2} + \cdots + \mathscr{D}^{i-1} \cup \mathscr{D}^1$, a result due to Keith. This gives an alternative inductive definition of "i-derivations" which is meaningful for not-necessarily-commutative rings but which possibly differs from Nakai's for commutative rings other than purely inseparable field extensions. If $\varphi \in \mathscr{D}^i(K/k)$, then it follows from the foregoing that $\Delta^{i-1}\varphi \in \mathscr{D}^1 \cup \mathscr{D}^1 \cup \cdots \cup \mathscr{D}^1$ (i times); in particular, $\Delta^{i-1}\varphi$ is a derivation as a function of any single variable. To these results we here add the following:

THEOREM. *If K/k is a finite purely inseparable field extension, and if γ is the least integer such that $\mathscr{D}^\gamma = (\mathrm{End}_k K)_0$, then* $\dim_K(\mathscr{D}^\gamma/\mathscr{D}^{\gamma-1}) = 1$.

It will follow that $\Delta^{\gamma-1}\mathscr{D}^\gamma$ is a one-dimensional K-space any generator of which will be called the "fundamental form" of K/k.

2. **Proof of the theorem.** An approximate automorphism of order m ("higher derivation" in the terminology of Jacobson [2]) of a not-necessarily-commutative k-algebra A is a formal polynomial $\Phi_t = 1 + t\varphi_1 + \cdots + t^m\varphi_m$ with $\varphi_i \in \mathrm{End}_k A$ ($1 = \mathrm{id}^A$) such that

$$\Phi_t(ab) = \Phi_t a \cdot \Phi_t b \bmod t^{m+1}$$

for all $a, b \in A$. That is, Φ_t is an automorphism of $A[t]/t^{m+1}$ (cf. [1]). This is equivalent to having

$$\delta\varphi_i = \varphi_i \cup \varphi_{i-1} + \varphi_2 \cup \varphi_{i-2} + \cdots + \varphi_{i-1} \cup \varphi_1, \qquad i = 1, \cdots, m.$$

It follows that φ_i is an i-derivation under the inductive definition valid for noncommutative rings. Those i-derivations which appear in approximate automorphisms will be called "special". If $\tilde{k}$ is an extension of the field k and $\tilde{A} = \tilde{k} \otimes_k A$ then $\mathscr{D}^i(\tilde{A}) = \tilde{k} \otimes_k \mathscr{D}^i(A)$, but the analogous assertion is meaningless for special i-derivations since the latter in general do not even form an additive group.

Suppose now that k has characteristic $p > 0$ and that $A = k[x]/(x^q - \alpha)$ where $q = p^e$ for some $e > 0$ and α is some element of k. Then A has an approximate automorphism Φ_t of order $q - 1$ which is completely defined by setting $\Phi_t x = x + t$. This implies that

$$\Phi_t x^m = x^m + \binom{m}{1} x^{m-1}t + \binom{m}{2} x^{m-2}t^2 + \cdots + t^m,$$

so writing $\Phi_t = 1 + t\varphi_1 + \cdots + t^{q-1}\varphi_{q-1}$, it follows that φ_i is an i-derivation sending x^m to $\binom{m}{i}x^{m-i}$ for all $m \geqq 0$. It is convenient to denote this i-derivation formally by $D^i/i!$, where $D = d/dx$. If we include the case $i = 0$, then $\mathrm{End}_k A$ can be shown to be a free A-module having the $D^i/i!$, $i = 0, 1, \cdots, q - 1$, as a basis. Since $(\Phi_t)^p x = x + pt = x$, one has $(D^i/i!)^p = 0$ for all $i > 0$. (Writing $i = i_0 + i_1 p + i_2 p^2 + \cdots + i_s p^s$ with $0 \leqq i_0, i_1, \cdots, i_s \leqq p - 1$, one has

$$D^i/i! = cD^{i_0}(D^p/p!)^{i_1}(D^{p^2}/p^2!)^{i_2} \cdots (D^{p^s}/p^s!)^{i_s}$$

where c is an integer $\not\equiv 0 \bmod p$.) The highest order of any derivation in $\operatorname{End}_k A$ is therefore $q - 1$, which is achieved by $D^{q-1}/(q-1)!$.

Let K/k be a finite purely inseparable field extension. By Pickert [6] (cf. also Rasala [7]) there is an extension $\tilde{k}$ (in fact there is a minimal finite one) such that writing $\tilde{A} = \tilde{k} \otimes_k A$ we have

$$\tilde{A} \cong \tilde{k}[x_1]/x_1^{q_1} \otimes \cdots \otimes \tilde{k}[x_r]/x_r^{q_r}$$

where $q_1 = p^{e_1}, \ldots, q_r = p^{e_r}$ for some $e_1, \ldots, e_r > 0$. Denoting the tensor factors of $\tilde{A}$ by $\tilde{A}_1, \ldots, \tilde{A}_r$, we have $\operatorname{End}_k \tilde{A} = \operatorname{End}_k \tilde{A}_1 \otimes \cdots \otimes \operatorname{End}_k \tilde{A}_r$, from which it follows that $\operatorname{End}_k \tilde{A}$ is generated by 1 and the various $D_l^i/i!$. (Therefore $\bigcup \mathscr{D}^i(\tilde{A}) = (\operatorname{End}_k \tilde{A})_0 = \tilde{k} \otimes (\operatorname{End}_k K)_0$, whence $\bigcup \mathscr{D}^i(K) = (\operatorname{End}_k K)_0$. This concise proof of Nakai's result is due to Keith.) The highest order achieved by any derivation in $\operatorname{End}_k \tilde{A}$ is that of

$$D_1^{q_1-1}/(q_1-1)! \otimes \cdots \otimes D_r^{q_r-1}/(q_r-1)!,$$

whose order is $\gamma = (q_1 - 1) + \cdots + (q_r - 1)$. Therefore $\tilde{\mathscr{D}}^\gamma = \mathscr{D}^\gamma(\tilde{A}) = (\operatorname{End}_k \tilde{A})_0$, and $\tilde{\mathscr{D}}^\gamma/\tilde{\mathscr{D}}^{\gamma-1}$ is a free A-module of rank 1. It follows that $\mathscr{D}^\gamma = \mathscr{D}^\gamma(K) = (\operatorname{End}_k K)_0$ and that $\dim_K(\mathscr{D}^\gamma/\mathscr{D}^{\gamma-1}) = 1$, as asserted by the theorem.

3. **Symmetric cup products, conjectures.** If f is a symmetric m-cochain and g a symmetric n-cochain of the k-algebra A with coefficients in itself, then we define the symmetric $m + n$-cochain $f * g$ by setting

$$(f * g)(a_1, \ldots, a_{n+m}) = (m!n!)^{-1} \sum f(a_{\sigma 1} \cdots a_{\sigma m})g(a_{\sigma(m+1)} \cdots a_{\sigma(m+n)})$$

where the sum is taken over all permutations of $1, \cdots, m + n$. This is meaningful regardless of the characteristic. One can verify that if $A = k[x]/(x^q - \alpha)$ then the fundamental form of A can be defined and equals $D \cup D \cup \cdots \cup D$ ($q - 1$ times), and that if we have a tensor product of such algebras, $A_1, \ldots, A_r$ with fundamental forms $f_1, \ldots, f_r$, then the fundamental form of $A_1 \otimes \cdots \otimes A_r$ is $f_1 * \cdots * f_r$. (This is always a coboundary if $r > 1$.) It follows that if a purely inseparable field extension K/k is of the form $K_1 \otimes_k K_2$, and if the fundamental forms of the factors are f_1 and f_2, then that of K is $f_1 * f_2$. We conjecture conversely that if the fundamental form factors then K is a tensor product. This has been shown if one puts certain stringent additional conditions on the factors, but the general question is open.

We remark finally that the "exponents" $e_1, \ldots, e_r$ of K/k can be determined once $\dim_K \mathscr{D}^i$ is known for $i = 1, \ldots, \gamma$, and these in turn depend on the Nakai operator Δ. For $\dim \mathscr{D}^i/\mathscr{D}^{i-1} = \dim \mathscr{D}^i - \dim \mathscr{D}^{i-1}$ is the number of ways of writing $i = i_1 + \cdots + i_r$ with $0 \leqq i_l \leqq q_l - 1$ $(= p^{e_l} - 1)$ for $l = 1, \ldots, r$. That is, it is the coefficient of t^i in

$$F(t) = \prod_{l=1}^{r} \frac{1 - t^{q_l}}{1 - t}.$$

Thus, knowing Δ determines $F(t)$, from which the $q_l = p^{e_l}$ can be determined.

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Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104