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FILTERED AND ASSOCIATED GRADED RINGS

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1. **Introduction.** The object of this note is to present a condition which guarantees that a filtered ring A is isomorphic (in the category of filtered rings) to its associated graded ring gr A. The result is that a separated, complete, nonnegatively filtered ring A over a field k of characteristic 0 is isomorphic to gr A if and only if $\dim_k H^2(\operatorname{gr} A, \operatorname{gr} A) = \dim_k H^2(A, A)$ where the $\dim_k H^2(\operatorname{gr} A, \operatorname{gr} A)$ is finite. The tool is algebraic deformation theory. Rim has observed that an application of the main theorem yields a condition for a plane algebroid curve over an algebraically closed field of characteristic 0 to be of the form $u^m = v^n$—a result obtained by Zariski [**5**] by a different approach.

2. Since A is a deformation of gr A (Gerstenhaber [**1**]), there exists a one-parameter family of deformations $A_t = \operatorname{gr} A[[t]]$ with multiplication defined by $f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots$. It is known that the deformation from gr A to A given by A_t is a "pop deformation", i.e., for $t \neq 0$, A_t is isomorphic as a filtered ring to $A[[t]]$ (Gerstenhaber [**2**]).

Let δ_t denote the Hochschild coboundary operator of the algebra A_t, i.e., computed relative to the multiplication f_t. For example, for $\varphi \in C^1(A_t, A_t)$, the group of 1-cochains of A_t, one has

$$\delta_t\varphi(a, b) = f_t(a, \varphi b) - \varphi(f_t(a, b)) + f_t(\varphi a, b).$$

If there exists $\eta_t \in C^1(A_t, A_t)$ such that $z_t = \delta_t\eta_t$, then $z_t \in B^2(A_t, A_t)$. $z_0 \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ is *extendible* if there exists $z_t \in Z^2(A_t, A_t)$ such that

$$z_t = z_0 + tz_1 + t^2z_2 + t^3z_3 + \cdots.$$

Note that every $b_0 \in B^2(\operatorname{gr} A, \operatorname{gr} A)$ is extendible since $b_0 = \delta\eta_0$ implies that $b_t = \delta_t\eta_0 = b_0 + tb_1 + t^2b_2 + \cdots$ is an extension of b_0 where η_0 is extended linearly over $k((t))$. An *extendible class* of $H^2(\operatorname{gr} A, \operatorname{gr} A)$ is a $[z_0]$ for which there is a representative z_0 which is extendible. $z_0 \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ is a *jump cocycle* if there exists an extension z_t of z_0 such that $z_t \in B^2(A_t, A_t)$. Each $b_0 = \delta\eta_0 \in B^2(\operatorname{gr} A, \operatorname{gr} A)$ is a jump cocycle since $b_t = \delta_t\eta_0$ is an extension of b_0 and $b_t \in B^2(A_t, A_t)$. A *jump class* of $H^2(\operatorname{gr} A, \operatorname{gr} A)$ is a $[z_0]$ for which there exists a representative z_0 which is a jump cocycle.

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[1] The results announced here are contained in the author's Ph.D. thesis. written under the guidance of Murray Gerstenhaber at the University of Pennsylvania.

The following theorem is the algebraic analogue of results obtained by Griffiths [3] for normed complexes and for fibered complex-analytic varieties. We assume the vector space dimension, $\dim_k H^2(\text{gr } A, \text{gr } A)$, is finite.

THEOREM 1.

$$\dim_{k((t))} H^2(A_t, A_t) = \dim_k \frac{\textit{Extendible classes of } H^2(\text{gr } A, \text{gr } A)}{\textit{Jump classes of } H^2(\text{gr } A, \text{gr } A)}$$

$$= \dim_k E/J.$$

PROOF. To prove that $\dim_{k((t))} H^2(A_t, A_t) \leqq \dim_k E/J$ one shows that a basis $[z_t^i], i = 1, \ldots, m$, of $H^2(A_t, A_t)$ over $k((t))$ can be chosen so that $z_t^i = z_0^i + tz_1^i + t^2 z_2^i + \cdots, [z_0^i]$ are linearly independent over k and $\{[z_0^i]\}$, the coset of $[z_0^i]$ in E/J, are linearly independent over k. The map $[z_t^i] \to \{[z_0^i]\}$ then establishes this inequality. The map: Extendible classes $\to H^2(A_t, A_t)$ defined by $[z_0^i] \to [z_t^i]$ has kernel equal to the jump classes. An elementary argument shows that this map $E/J \to H^2(A_t, A_t)$ preserves linear independence. Thus $\dim_k E/J \leqq \dim_{k((t))} H^2(A_t, A_t)$.

The multiplication of A_t has been defined as $f_t(a, b) = ab + tF_1(a, b) + t^2 F_2(a, b) + \cdots$.

PROPOSITION 1. *F_1 is extendible.*

PROOF. Define $F_t(a, b) = F_1(a, b) + 2tF_2(a, b) + 3t^2 F_3(a, b) + \cdots$. F_t is an extension of F_1 since $f_t(a, f_t(b, c)) - f_t(f_t(a, b), c) = 0$ holds and the formal derivative of this is

$$f_t(a, F_t(b, c)) + F_t(a, f_t(b, c)) - F_t(f_t(a, b), c) - f_t(F_t(a, b), c) = 0$$

which is precisely the condition for F_t to be a δ_t-cocycle.

It is important, as Rim observes, that F_t, the derivative of the multiplication f_t, not only is a cocycle of the deformed algebra but is actually intrinsic to the deformed algebra and represents a cohomology class which would not be altered if f_t were replaced by an equivalent multiplication g_t. This is proved by the following observations. If f_t and g_t are equivalent multiplications, then

$$(1) \qquad f_t(a, b) = \psi_t^{-1}(g_t(\psi_t a, \psi_t b))$$

where ψ_t is a linear automorphism. The formal derivative of $\psi_t f_t(a, b) = g_t(\psi_t a, \psi_t b)$ is

$$(2) \quad \psi_t'(f_t(a, b)) + \psi_t F_t(a, b) = G_t(\psi_t a, \psi_t b) + g_t(\psi_t' a, \psi_t b) + g_t(\psi_t a, \psi_t' b)$$

where G_t is the derivative of g_t. From (1) and (2) it follows that

$$\psi_t^{-1}\psi_t'(f_t(a, b)) + F_t(a, b)$$
$$= \psi_t^{-1} G_t(\psi_t a, \psi_t b) + f_t(\psi_t^{-1}\psi_t' a, b) + f_t(a, \psi_t^{-1}\psi_t' b).$$

Therefore $F_t(a, b) = \psi_t^{-1}G_t(\psi_t a, \psi_t b) + \delta_t\psi_t^{-1}\psi_t'(a, b)$ where δ_t is defined with respect to f_t multiplication and the cohomology class in $H^2(A_t, A_t)$ determined by F_t is not altered by a change of basis.

PROPOSITION 2. *F_1 is a jump cocycle.*

PROOF. Let Φ_t be an algebra isomorphism of A_t onto A_1 where $t \neq 0$. Then $\Phi_t(f(a, b)) = f_1(\Phi_t a, \Phi_t b)$ and the derivative of both sides of this expression is

$$\Phi_t'(f_t(a, b)) + \Phi_t(F_t(a, b)) = f_1(\Phi_t' a, \Phi_t b) + f_1(\Phi_t a, \Phi_t' b)$$

where Φ_t' is the formal derivative of Φ_t. Rewriting this expression yields

$$F_t(a, b) = f_t(\Phi_t^{-1}\Phi_t' a, b) - \Phi_t^{-1}\Phi_t'(f_t(a, b)) + f_t(a, \Phi_t^{-1}\Phi_t' b)$$

and thus $F_t = \delta_t\Phi_t^{-1}\Phi_t'$.

THEOREM 2. *A separated, complete filtered ring A over a field k of characteristic 0 is isomorphic to* $\mathrm{gr}\, A$ *if and only if* $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_k H^2(A, A)$ *where the vector space* $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$ *is finite.*

PROOF. By [2], $A[[t]]$ is isomorphic to A_t for $t \neq 0$. The

$$\dim_{k((t))} H^2(A[[t]], A[[t]]) = \dim_k H^2(A, A).$$

It is therefore sufficient to prove that, for $t \neq 0$, A_t is isomorphic to $\mathrm{gr}\, A[[t]]$ with multiplication f_0 if $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_{k((t))} H^2(A_t, A_t)$. By Theorem 2,

$$\begin{aligned}\dim_{k((t))} H^2(A_t, A_t) &= \dim_k \frac{\textit{Extendible classes of } H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{\textit{Jump classes of } H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)} \\ &\leqq \dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A).\end{aligned}$$

Therefore the $\dim_{k((t))} H^2(A_t, A_t) = \dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$ implies that the jump classes of $H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \{\text{coboundaries}\}$. But F_1 is a jump cocycle. Thus $F_1 = \delta\rho_1$ and $P_t(a) = a - t\rho_1(a)$ is an isomorphism of A_t to $\mathrm{gr}\, A[[t]]$ with multiplication $ab + t^2F_2(a, b) + t^3F_3(a, b) + \cdots$. Provided k has characteristic 0 the above argument can be repeated for F_2 and, in general, for F_n to show that F_n is a jump cocycle with the derivative of the appropriate multiplication taken as the extension of F_n. By the assumption on dimension, $F_n = \delta\rho_n$. Therefore A_t is isomorphic to $\mathrm{gr}\, A[[t]]$ with multiplication f_0.

3. Let k be an algebraically closed field of characteristic 0 and let $f(x, y)$ be an irreducible power series with coefficients in k. Let C be the plane curve defined by $f = 0$, A be the local ring of C and m_A be the maximal ideal of A.

The Weierstrass Preparation Theorem and Puiseux's Theorem together imply that $A \subset k[[t]]$. Thus we can define a filtration on A so that $F_0A = A \supset F_1A = t \cap A \supset F_2A = t^2 \cap A \supset \cdots$ and form the associated graded ring gr A. We may assume $\mathrm{gr}_1 A = F_1A/F_2A = 0$ since otherwise $t \in A$ implies that $A = k[[t]]$ and the curve C would be nonsingular.

gr $A = k[[t^{v_1}, t^{v_2}, \ldots t^{v_r}]]$ with $v_1 < v_2 < \cdots < v_r$ by definition of the filtration on A. Since A is the local ring of a plane algebroid curve, gr A is generated by at most two elements.

The main result of §1 states that A is isomorphic to gr A if and only if $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_k H^2(A, A)$. Thus a suitable basis $\{u, v\}$ of m_A can be chosen so that the curve C is of the form $u^m = v^n$ provided that $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_k H^2(A, A)$. Rim has observed that these results give an alternate form to a result of Zariski [5] which states that $l(T) = L$ if and only if for a suitable basis $\{x, y\}$ of m_A the equation of the curve C is of the form $y^n = x^m$ where $(n, m) = 1$ by the irreducibility of the curve C, $l(T)$ is the length of the A-module T (T is the torsion submodule of the module of Kähler differentials of A) and L is the length of the conductor of A in the integral closure $\bar{A}$ of A.

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