BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

LOCALIZATION AND COMPLETION

BY J. LAMBEK

Communicated by Alex Rosenberg, December 27, 1971

Let I be an injective right R-module and E its ring of endomorphisms. The functor

$$\mathrm{Hom}_R(-, I): \mathrm{Mod}\, R \to (E\, \mathrm{Mod})^{\mathrm{op}}$$

has a right adjoint $\mathrm{Hom}_E(-, I)$, giving rise to a triple (standard construction) (S, η, μ), where

$$S = \mathrm{Hom}_E(\mathrm{Hom}_R(-, I), I).$$

Let $Q \to S$ be the equalizer of the pair of maps $\eta S, S\eta: S \to S^2$, then Q is the bext co-approximation of S by an idempotent triple [**2**].

$Q(M)$ is called the *localization* or *module of quotients* of M at I, and $Q(R)$ is a ring, the *ring of quotients*. $Q(M)$ may also be described as the divisible hull of M made torsionfree, in the *torsion theory* obtained from I (see e.g. [**4**]). In this torsion theory one calls M *torsion* if $\mathrm{Hom}_R(M, I) = 0$, *torsionfree* if M is isomorphic to a submodule of a power of I, *divisible* if $I(M)/M$ is torsionfree, where $I(M)$ is the injective hull of M.

The endofunctor Q of $\mathrm{Mod}\, R$ is always left exact. It is isomorphic to the identity functor if and only if it may be obtained from

$$I = \mathrm{Hom}_R(F, \boldsymbol{Q}/\boldsymbol{Z}),$$

where F is a free left R-module. Q is isomorphic to $(-) \otimes_R Q(R)$ if and only if it may be obtained from $I = \mathrm{Hom}_R(F, \boldsymbol{Q}/\boldsymbol{Z})$, where $R \to F$ is an epimorphism of rings and F is a flat left R-module [**4**], [**7**], [**10**]. Q is exact if and only if $I(M)/M$ is divisible for all torsionfree divisible modules M (compare with [**3**]), and then $Q(M) \cong M \otimes_R Q(R)$ for every finitely presented module M. We also note that every divisible module is injective if and only if I has zero singular submodule.

The I-adic topology on M is defined by taking as fundamental open neighborhoods of zero all kernels of homomorphisms $M \to I^n$, where n is finite. This topology and its relation to the usual P-adic topology has been discussed in [**5**].

The torsion submodule of M is open in the I-adic topology if and only if $\mathrm{Hom}_R(M, I)$ is a finitely generated E-module, or, equivalently, $\mathrm{Hom}_R(M, I^n)$ is a principal $\mathrm{End}_R(I^n)$-module for some finite n. These

AMS 1970 *subject classifications*. Primary 16A08; Secondary 18C15.

conditions imply that $Q(M) = S(M)$. Since $S(R)$ is the bicommutator of I, one obtains known results by Morita and the author as special cases.

While $Q(M)$ is a closed submodule of $S(M)$ in the I-adic topology, one obtains a density theorem when $S(M)$ is endowed with the *finite* topology: the topology induced by the product topology of $I^{\mathrm{Hom}_R(M,I)}$ when I is discrete.

THEOREM. *When Q is exact, $S(M)$ with the finite topology is the completion of $Q(M)$ with the I-adic topology.*

For $M = R$, this specializes to a known result [**5**] about the bicommutator of I.

The proof of this theorem makes use of certain algebras and homomorphisms of the triple (S, η, μ) in the sense of [**1**]. When Q is exact, the algebras of this triple may be characterized as torsionfree divisible modules equipped with a *limit operation* λ which assigns a limit to each I-adic Cauchy net. One requires that λ is R-linear, that it sends every convergent net to its usual limit, and that the limit of a product net may be computed as an interated limit.

REFERENCES

0. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass., 1969. MR **39** # 4129.
1. S. Eilenberg and J. C. Moore, *Adjoint functors and triples*, Illinois J. Math. **9** (1965), 381–398. MR **32** # 2455.
2. S. Fakir, *Monade idempotente associeé à une monade*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A99-A101. MR **41** # 1828.
3. O. Goldman, *Rings and modules of quotients*, J. Algebra **13** (1969), 10–47. MR **39** # 6914.
4. J. Lambek, *Torsion theories, additive semantics, and rings of quotients*, Lecture Notes in Math., no. 177, Springer-Verlag, Berlin and New York, 1971.
5. ———, *Bicommutators of nice injectives*, J. Algebra **21** (1972), 60–73.
6. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. MR **20** # 5800.
7. K. Morita, *Flat modules, injective modules and quotient rings*, Math. Z. **120** (1971), 25–40.
8. B. J. Müller, *Linear compactness and Morita duality*, J. Algebra **16** (1970), 60–66. MR **41** # 8474.
9. B. Rattray, *Non-additive torsion theories* (to appear).
10. B. Stenström, *A survey of the theory of rings of quotients*, Lecture Notes in Math., no. 237, Springer-Verlag, Berlin and New York, 1971.

DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, CANADA