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# A REPRESENTATION OF A POSITIVE LINEAR MAPPING

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Let $X$ and $Y$ be compact Hausdorff spaces. Let $C(X)$ and $C(Y)$ be the algebras of real valued continuous functions on $X$ and $Y$ respectively. $C(X)$ and $C(Y)$ are endowed with their natural partial ordering and their sup norm. Let $\Phi: C(X) \to C(Y)$ be a positive, bounded linear mapping.

$X$ is said to have the Souslin property if every disjoint family of non-empty open subsets of $X$ is countable.

A lattice $L$ is said to satisfy the countable chain condition upward if the following is true: For any upper bounded subset $A$ of $L$, there exists a countable subset $B$ of $A$ such that $A$ and $B$ have the same family of upper bounds. The countable chain condition downward on a lattice can be defined in a similar fashion.

A lattice $L$ is said to satisfy the countable chain condition if $L$ satisfies both the countable chain condition upward and the countable chain condition downward.

The purpose of this note is to announce the results on representation for $\Phi$, based on the techniques developed in [**1**], [**2**].

To get the main theorem, we need the following series of propositions which are interesting in themselves.

PROPOSITION. *For a given compact Hausdorff space $X$, there exists a complete Boolean space $X^*$ and a mapping $\sigma: C(X) \to C(X^*)$ such that $\sigma$ is an isometric, order preserving and algebra monomorphism.*

REMARK. The construction of $\sigma$ here is different from the one in [**3**]. A part of the proof comes from an application of the Gelfand-Naimark theorem [**4**].

We study a necessary and sufficient condition on $X$ under which $C(X^*)$ satisfies the countable chain condition so that we later use this result to represent $\Phi$ as the Maharam integral [**2**].

To this end, we introduce the concept of the countable chain condition on a Boolean algebra [**6**] and the pseudocountable chain condition on $C(X)$.

$C(X)$ is said to satisfy the pseudocountable chain condition if every disjoint set of nonzero elements of $C(X)$ is countable. (Two functions $f$ and $g$ of $C(X)$ are disjoint if $\inf(f, g) = 0$.)

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PROPOSITION. *$X$ has the Souslin property if and only if $C(X^*)$ has the countable chain condition.*

REMARK. The proof goes roughly as follows: First we show that the countable chain condition on $C(X^*)$, the pseudocountable chain condition on $C(X^*)$ and the Souslin property on $X^*$ are all equivalent. Next, we show that $X^*$ has the Souslin property if and only if $X$ has the Souslin property.

We are concerned with an extension of $\Phi$. Let $K(X)$ and $K(Y)$ be the spaces of Baire functions on $X$ and $Y$ respectively. In [5], it was shown that $K(X)$ and $K(Y)$ contain $C(X)$ and $C(Y)$ respectively.

PROPOSITION. *There is a unique extension $\Phi_1: K(X) \to K(Y)$ of $\Phi$ with $\|\Phi_1\| = \|\Phi\|$. Furthermore, $\Phi_1$ is a positive, linear and countably additive mapping.*

Finally, we have the following theorem.

THEOREM. *Let $X$ and $Y$ have the Souslin property. Then $\Phi$ can be expressed as the Maharam integral.*

REMARK 1. For the definition of the Maharam integral, we refer to [2]. Roughly, we may rephrase the theorem as follows. Under the above assumptions on $X$ and $Y$, there exist compact Hausdorff spaces $R$ and $S$ such that $C(X^*)$ is "isomorphic" to a certain space of functions on $R \times S$ and $C(Y^*)$ is isomorphic to a space of functions on $R$, and under these isomorphisms, $\Phi$ corresponds to the mapping $f \mapsto f'$ where $f'(r) = \int_S f(r, s)\, d\mu$, the integral being formed with respect to an ordinary $\sigma$-finite numerical measure $u$ on $S$.

REMARK 2. The proof relies on the preceding propositions and the techniques e.g., a direct product $J \otimes U$ in Maharam's sense, developed in [1], [2] to get a generalized form of the Maharam integral. To complete the proof, it is necessary to realize a certain set mapping as a point mapping.

Detailed proofs and applications of these results will appear elsewhere.

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