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MAJORIZATION-SUBORDINATION THEOREMS FOR LOCALLY UNIVALENT FUNCTIONS

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ABSTRACT. We generalize three different quantitative majorization-subordination theorems of classical univalent function theory due to Goluzin, Lewandowski, MacGregor and Tao Shah to linear invariant families of locally univalent analytic functions of finite order.

Let S denote the family of functions $f(z)$ which are analytic and univalent in the open unit disc D and which are normalized so that $f(0) = 0$, $f'(0) = 1$.

Let $f(z)$, $F(z)$ and $\varphi(z)$ be functions analytic in $|z| < r$. We say that $f(z)$ is *majorized* by $F(z)$ in $|z| < r$ if $|f(z)| \leqq |F(z)|$ in $|z| < r$. We say that $f(z)$ is *subordinate* to $F(z)$ in $|z| < r$ if $f(z) = F(\varphi(z))$ where $|\varphi(z)| \leqq |z|$ *in* $|z| < r$. We denote majorization and subordination in $|z| < r$ by $f < < F$ and $f \prec F$ respectively. Equivalently, $f(z)$ is majorized by $F(z)$ in $|z| < r$ if and only if $f(z) = \varphi(z)F(z)$ where $|\varphi(z)| \leqq 1$ in $|z| < r$ [**5**, Lemma 1]. Also, if $F(z)$ is locally univalent in $|z| < r$, then $f(z)$ is subordinate to $F(z)$ in $|z| < r$ if and only if for each fixed $R < r$, the image of the disc $|z| \leqq R$ under $f(z)$ is contained in its image under $F(z)$ [**4**, p. 163].

Biernacki [**1**] in 1936 obtained the first results of majorization-subordination theory. He showed that if $F(z) \in S$ and $f(z) \prec F(z)$ in D, then $f(z) < < F(z)$ in $|z| < \frac{1}{4}$. Goluzin [**2**, p. 376] improved the result and Tao Shah [**9**] in 1957 obtained the complete solution for S by showing that $f(z) < < F(z)$ in $|z| < (3 - \sqrt{5})/2$ and that the result is best possible. Subsequent investigations have focused on three problems. Let $F(z)$ be an arbitrary function in S.

(1) If $f(z) < < F(z)$ in D, determine the largest r for which $f'(z) < < F'(z)$ in $|z| < r$ MacGregor [**5**].

(2) If $f(z) \prec F(z)$ in D, determine the largest r for which $f'(z) < < F'(z)$ in $|z| < r$ Goluzin [**2**].

(3) If $f(z) < < F(z)$ in D, determine the largest r for which $f(z) \prec F(z)$ in $|z| < r$ Lewandowski [**3**].

We generalize each of the above three problems by allowing $F(z)$ to be in U_α, the universal linear invariant family of order α. We have found that the radius r in each question is a function of the order α. Furthermore, each of the known results for S is a special case of our results for the particular family U_2. Since the families U_α, $\alpha > 1$, contain functions of *infinite*

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valence, our investigations show that the important datum for majorization-subordination theory is not univalence, but the order of a linear invariant family. Since the radius r in each question is a function of the order, our results explain why such constants as $2-\sqrt{3}$, $3-\sqrt{5}$, etc. appear in univalent function theory; they are reflecting the fact that the order of the linear invariant family S is precisely 2.

Following Pommerenke [7], a family of functions M is said to be *linear invariant* if the following two conditions are satisfied:

(1) all functions $f(z)$ in M are analytic and locally univalent (that is, $f'(z) \neq 0$) in D and have the form $f(z) = z + a_2z^2 + \cdots$;

(2) if $\varphi(z)$ is a bilinear map of D onto D and $f(z)$ belongs to M, then the function

$$\Lambda_\varphi[f(z)] = [f(\varphi(z)) - f(\varphi(0))]/\varphi'(0)f'(\varphi(0)) = z + \cdots$$

must also belong to M. The *order* of a linear invariant family M is defined as

$$\alpha = \sup_{f \in M} |f''(0)/2|.$$

The order of a family is always $\geqq 1$ [7, p. 117].

Let U_α $(1 \leqq \alpha < \infty)$ be the universal linear invariant family of order α; that is, the union of all linear invariant families of order $\leqq \alpha$. The family U_1 is precisely the family of all normalized convex univalent functions [7, p. 134]. Close-to-convex functions of order β are in $U_{\beta-1}$ [8, p. 182]. Functions whose boundary rotation is bounded by A are in U_α where $\alpha = A/2\pi$. The family S is in U_2 [7, p. 115]. Each family U_α $(\alpha > 1)$ contains functions of infinite valence.

MacGregor investigated the effect that majorization by a univalent function has on the radius of majorization of the derivatives. We have not only obtained corresponding results for majorization by a function in U_α but a simplified proof that the result is sharp.

THEOREM 1. *Let $f(z)$ be majorized by $F(z)$ in D. If $F(z) \in U_\alpha$, $1 \leqq \alpha < \infty$, then $f'(z)$ is majorized by $F'(z)$ in*

$$|z| \leqq [(\alpha+1)^{1/\alpha} - 1]/[(\alpha+1)^{1/\alpha} + 1] = \tanh[(2\alpha)^{-1}\ln(\alpha+1)].$$

The result is best possible for each α.

COROLLARY 1. *Let $f(z) < < F(z)$ in D. If $F(z) \in U_1$, then $f'(z) < < F'(z)$ in $|z| \leqq \frac{1}{3}$. If $F(z) \in U_2$, then $f'(z) < < F'(z)$ in $|z| \leqq 2-\sqrt{3}$.*

Corollary 1 not only yields MacGregor's result for convex univalent functions (U_1), but, since S is a proper subset of U_2, also yields a strengthening of MacGregor's Theorem 1, part B.

Lewandowski [3] in 1961 showed that majorization by a univalent function forced subordination in $|z| < .21$ and thus established a converse to the Biernacki problem. The following theorem gives upper and lower estimates for the radius of subordination when the majorizing function is only assumed to be locally univalent.

THEOREM 2. *Let $f(z)$ be majorized by $F(z)$ in D with $f'(0) \geqq 0$. Let $R_1(\alpha)$ be the root in* $[0, 1]$ *of the equation*

$$\frac{2x}{1+x^2} - \left(\frac{1-x}{1+x}\right)^{\alpha}\left(1 - \frac{1}{4}\left(\frac{1-x}{1+x}\right)^{2\alpha}\right)^{1/2} = 0$$

and let $R_2(\alpha)$ be the root in $[0, 1]$ *of the equation*

$$x(1+x)^{\alpha} - (1-x)^{\alpha} = 0.$$

If $F(z) \in U_\alpha$, $1 \leqq \alpha \leqq 2.88$, then $f(z)$ is subordinate to $F(z)$ in $|z| < R(\alpha)$ where $R_1(\alpha) \leqq R(\alpha)$. Furthermore, for all $\alpha \geqq 1$, $R(\alpha) \leqq R_2(\alpha)$.

COROLLARY 2. *Let $f(z) < < F(z)$ with $f'(0) \geqq 0$. If $F(z) \in U_1$, then $f(z) \prec F(z)$ in $|z| < R$, where $.28 < R \leqq \sqrt{2} - 1$. If $F(z) \in U_2$, then $f(z) \prec F(z)$ in $|z| < R$ where $.21 < R < .3$.*

Corollary 2 generalizes Lewandowski's result and appears to be a new result for the convex univalent functions.

Goluzin in 1951 showed that subordination by a univalent function forced majorization of the derivatives in $|z| < .12$. Tao Shah [**10**] in 1957 improved Goluzin's results and showed that majorization of the derivatives occurred inside $|z| < 3 - \sqrt{8}$ and the result is best possible. We have established

THEOREM 3. *Let $f(z)$ be subordinate to $F(z)$ in D. If $f'(0) \geqq 0$ and $F(z) \in U_\alpha$, $1.65 \leqq \alpha < \infty$, then $f'(z)$ is majorized by $F'(z)$ in*

$$|z| \leqq \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}.$$

The inequality is sharp.

We obtain as an immediate corollary a strengthening of Tao Shah's original theorem.

COROLLARY 3. *If $f(z) \prec F(z)$ in D, $f'(0) \geqq 0$ and $F(z) \in U_2$, then $f'(z) < < F'(z)$ in $|z| < 3 - \sqrt{8}$.*

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