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NONCOBORDANT FOLIATIONS OF S^3

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In this note, we will sketch the construction of uncountably many noncobordant foliations of S^3, and a surjective homomorphism $\pi_3(B\Gamma_1^r) \twoheadrightarrow R$ $[2 \leqq r \leqq \infty]$, where $B\Gamma_1^r$ is the classifying space for singular C^r codimension one foliations constructed by Haefliger ([3], [4]).

Godbillon and Vey have recently discovered certain cohomology classes associated with foliations, and more generally, with Γ_p^r-structures or singular C^r codimension p foliations [2]. The cohomology invariant Γ_F is defined very simply for a codimension one, transversely oriented foliation F determined by a C^2 one-form ω. The condition of integrability for a one-form is $d\omega \wedge \omega = 0$. Then for some one-form θ, $d\omega = -\theta \wedge \omega$. Γ_F is defined to be the deRham cohomology class of the closed form $\theta \wedge d\theta$. If F is not transversely oriented, Γ_F may be defined via two-sheeted covers. If F is C^2 but not given by a C^2 one-form, it is still possible to define Γ_F. Γ_F depends only on F, not on ω and θ, and is natural; so if $f : M \to N$, where N has foliation F and f induces a foliation f^*F on M, then $\Gamma_{f^*F} = f^*\Gamma_F$. It follows that $\Gamma_F[M^3]$ is an invariant of the cobordism class of F. That is, if $\partial N^4 = M_1 + -M_2$, and if N^4 has foliation F transverse to ∂N^4 inducing F_1 on M_1 and F_2 on M_2, then $\Gamma_{F_1}[M_1] = \Gamma_{F_2}[M_2]$.

The form $\theta \wedge d\theta$ may be interpreted as a measure of the helical wobble of the leaves of F, as in Figure 1. In order that the cohomology class Γ_F be nontrivial, there must be some kind of global phenomenon corresponding to helical wobble.

Now consider the hyperbolic plane H^2 and its unit tangent bundle $T_1(H^2)$. There is a foliation F of $T_1(H^2)$ invariant under the isometries of H^2: each leaf of F consists of the forward unit tangents to a family of parallel geodesics. In non-Euclidean geometry, parallel means asymptotic
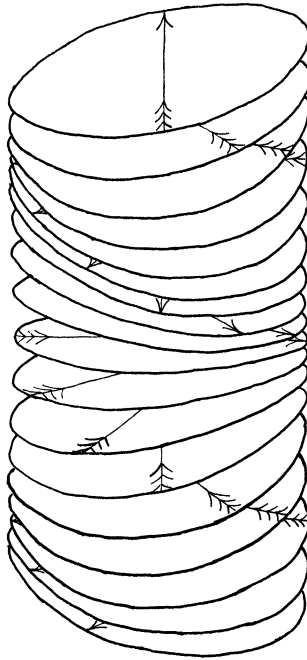

FIGURE 1. *Helical wobble of the leaves of a foliation.* The disks here have a constant angle of inclination to the central axis, in a direction which rotates at a constant rate. Their centers are evenly spaced. The arrows are perpendicular to the central axis, and they represent the direction in which the leaves are most squeezed together.

in the positive direction. The foliation F is transverse to the fibres of $T_1(H^2)$.

Let P be any convex polygon in H^2. We will construct a foliation of the three-sphere S^3 depending on P. Let the sides of P be labelled $s_1, \ldots, s_k$, and let the angles have magnitudes $\alpha_1, \ldots, \alpha_k$. Let Q be the closed region bounded by $P \cup P'$, where P' is the reflection of P through s_1. Let Q_ε be Q minus an open ε-disk about each vertex. If $p: T_1(H^2) \to H^2$ is projection, then $p^{-1}(Q)$ is a solid torus (with edges) with foliation F_1 induced from F. For each i, there is a unique orientation-preserving isometry of H^2, denoted I_i, which matches s_i point-for-point with its reflected image s_i'. We glue the cylinder $p^{-1}(s_i \cap Q_\varepsilon)$ to the cylinder $p^{-1}(s_i' \cap Q_\varepsilon)$ by the differential dI_i for each $i > 1$, to obtain a manifold $M = (S^2$ minus k punctures$) \times S^1$, and a glued foliation F_2 induced from F_1. With a little thought, we see that F_2 intersects each boundary component of M, T_i^2, transversely, and induces there a foliation analytically conjugate to a linear foliation of slope α_k/π. Now we spin each of these linear foliations around a torus leaf; then we glue in k Reeb components, the first one sideways so that we obtain S^3 with a smooth foliation F_P.

THEOREM. $\Gamma_{F_P}[S^3] = 4\pi \text{ Area}(P)$.

INDICATION OF PROOF. When the forms ω and θ defining F on $T_1(H^2)$ are invariant under the orientation-preserving isometries, $\theta \wedge d\theta$ is the volume form. The foliation F_P does not depend on ε; if ε is small, and if ω and θ defining F_P agree with the forms on $T_1(H^2)$ except in a small neighborhood of the Reeb components, they may be extended so the integral of $\theta \wedge d\theta$ is nearly 0 over the Reeb component neighborhoods. Then $\Gamma_{F_P}[S^3]$ approximates, therefore equals, the volume of $p^{-1}(Q)$.

COROLLARY. *There are uncountably many noncobordant C^∞ foliations of S^3. The Godbillon-Vey invariant is a surjective homomorphism of $\pi_3(B\Gamma_1^r)$* $[2 \leq r \leq \infty]$ *onto the reals.*

There are further examples of foliations which are analytic, transverse to the fibres of an S^1 bundle over a surface, and for which $\Gamma_F[M]$ takes all real values. From this follows the

THEOREM. $H_3(B\Gamma_1^\omega; Z)$ *and* $H_2(\text{Diff}_+^r(S^1); Z)$ $[2 \leq r \leq \omega]$ *have surjective homomorphisms onto the reals, where* $\text{Diff}_+^r(S^1)$ *is the group of orientation-preserving diffeomorphisms of S^1 of class C^r, with the discrete topology.*

REMARKS. The area of P is the hyperbolic deficiency of the sum of the angles of P, that is, $\text{Area}(P) = (k - 2)\pi - \sum_k \alpha_k$. Note how this information is retained in F_P. When the angles α_k all divide π, F_P may also be obtained from $T_1(H^2) = SL(2)$ modulo a discrete subgroup, with surgery performed along transverse curves. A similar construction could have been based on surfaces having a number of isolated corners, with metrics of constant negative curvature everywhere else.

These results contrast sharply with the C^0 case, for Mather has shown that $B\Gamma_1^0$ is a $K(Z_2, 1)$ [5]. It follows that the foliations F_P on S^3 bound C^0, but not C^2, Γ_1-structures on D^4.

In the analytic case, Haefliger has shown that $B\Gamma_1^\omega$ is a $K(\pi_1, 1)$ for some group π_1 which is uncountable and has a subgroup of index 2 which equals its commutator subgroup [4].

Gel'fand and Fuks [1] give the continuous Eilenberg-Mac Lane cohomology $H_c^*(\text{Diff}^\infty(S^1); R)$ [do they mean $\text{Diff}_+^\infty(S^1)$?] which in dimension two has two generators. The Euler class, and the Godbillon-Vey invariant integrated over the fibres of the natural S^1 bundle give two linearly independent classes in $H_c^2(\text{Diff}_+^\infty(S^1); R)$ which map to linearly independent classes in the ordinary Eilenberg-Mac Lane cohomology, but to zero in $H_c^2(\text{Diff}^\infty(S^1); R)$. Perhaps, using the work of Gel'fand and Fuks, it will be possible to compute the continuous Eilenberg-Mac Lane cohomology of Γ_P^∞; that is, the ring of cohomology invariants depending continuously on foliations.

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