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GENERALIZED PRODUCT THEOREMS FOR TORSION INVARIANTS WITH APPLICATIONS TO FLAT BUNDLES

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This note announces generalizations of the product theorems for Wall invariants and Whitehead torsions due to Gersten [**5**], Siebenmann [**7**, Chapter VII], and Kwun and Szczarba [**6**], and applies these theorems to study torsion invariants of the total space of a flat bundle. The generalized product theorems are described in §§1 and 2. The applications are found in §3.

These theorems were discovered in an attempt to understand more clearly the orientation phenomena discovered in [**1**] and [**2**] by concentrating attention on bundles in which "orientation" is a complete bundle invariant. The author would like to thank D. Sullivan whose use of the word "flat" in a private conversation stimulated this work.

0. **Basic algebraic definitions and notations.** Let R be a commutative ring with unit. (Usually $R = Z$, the ring of integers, or Q the rational numbers.) For any group π, $\mathfrak{P}R(\pi)$, and $\sum\mathfrak{P}R(\pi)$ will denote the category of finitely generated projective modules over $R(\pi)$, and the category with objects (P, f) with $P \in \mathfrak{P}R(\pi)$ and $f:P \to P$ and $R(\pi)$ isomorphism. A morphism $g:(P_1, f_1) \to (P_2, f_2)$ is an $R(\pi)$ homomorphism $g:P_1 \to P_2$ such that $f_2g = gf_1$.

$K_0R(\pi)$ and $K_1R(\pi)$ will be usual algebraic K-theoretic groups (cf. [**3**, pp. 344–348]). $[P]$ or $[P, f]$ will denote the class of P and (P, f) in $K_0R\pi$ and $K_1R\pi$ respectively. The quotient of $K_1R(\pi)$ by the subgroup $\pm\pi$ will be denoted Wh $R(\pi)$ and will be called the R-Whitehead group of π. When $R = Z$ this is the usual Whitehead group. If $j:\pi \to \pi'$ is a homomorphism, j_* will denote any of the induced maps on K_0, K_1, or Wh.

Let A and B be groups and $\alpha:B \to \operatorname{Aut} A$ be a homomorphism. Then $A \times_\alpha B$ will denote the semidirect product of A and B with respect to α. As sets $A \times_\alpha B = A \times B$. The multiplication on $A \times_\alpha B$ is given by $(a, b)(a', b') = (a\alpha(b)(a'), bb')$. The functions $k:A \to A \times_\alpha B$, $p:A \times_\alpha B \to B$, and $s:B \to A \times_\alpha B$ given by $k(a) = (a, 1)$, $p(a, b) = b$, and $s(b) = (1, b)$ are homomorphisms. α extends to a homomorphism, also denoted by $\alpha, \alpha:B \to \operatorname{Aut} R(A)$.

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1. **α-semilinear representations.** Let A, B, α be as above and $M \in \mathfrak{P}R(A)$. A representation $\rho: B \to \mathrm{Aut}_R M$ is *α-semilinear* (cf. [**4**]) if $\rho(b)(\lambda m) = \alpha(b)(\lambda)\rho(b)(m)$ for all $b \in B, \lambda \in R(A), m \in M$.

Let $\mathfrak{R}(R(B), R(A), \alpha)$ be the category with objects pairs (P, ρ) where $P \in \mathfrak{P}R(A)$ and $\rho: B \to \mathrm{Aut}_R P$ is α-semilinear. A morphism $f:(P_1, \rho_1) \to (P_2, \rho_2)$ in $\mathfrak{R}(R(B), R(A)\alpha)$ is an $R(A)$ map $f: P_1 \to P_2$ such that $f\rho_1(b) = \rho_2(b)f$ for every $b \in B$. The Grothendieck group of the category $\mathfrak{R}(R(B), R(A), \alpha)$ is called the *group of α-semilinear representations* of B over $R(A)$ and is denoted $G_R(B, A, \alpha)$. When $A = 1$, $G_R(B, A, \alpha) = G_R(B)$ is the usual group of representations of B over R.

THEOREM 1. *There are pairings*

$$T_0: G_R(B, A, \alpha) \otimes K_0 R(B) \to K_0 R(A \times_\alpha B),$$

$$T_1: G_R(B, A, \alpha) \otimes K_1 R(B) \to K_1 R(A \times_\alpha B).$$

SKETCH OF THE PROOF. The proof uses some ideas of Swan developed in [3, pp. 563–566]; namely, for T_0, define a functor

$$\mathfrak{T}_0: \mathfrak{R}(R(B), R(A), \alpha) \times \mathfrak{P}R(B) \to \mathfrak{P}R(A \times_\alpha B)$$

by setting $\mathfrak{T}_0((P, \rho), Q) = P \otimes_R Q$ and making $P \otimes_R Q$ into an $R(A \times_\alpha B)$ module by setting $(a, b)p \otimes q = a\rho(b)(p) \otimes bq$. By Frobenius reciprocity [3, p. 563], if P is free over $R(A)$ of rank m and Q is free over $R(B)$ of rank n, then $P \otimes_R Q$ is free of rank mn over $R(A \times_\alpha B)$. A direct sum argument then shows that if $P \in \mathfrak{P}R(A)$ and $Q \in \mathfrak{P}R(B)$, then $P \otimes_R Q \in \mathfrak{P}R(A \times_\alpha B)$. The proof is completed by noting that $\mathfrak{T}_0$ preserves short exact sequences in either variable.

The pairing T_1 is induced from the functor

$$\mathfrak{T}_1: \mathfrak{R}(R(B), R(A), \alpha) \times \textstyle\sum\mathfrak{P}R(B) \to \textstyle\sum\mathfrak{P}R(A \times_\alpha B),$$

given by $\mathfrak{T}_1((P, \rho), (Q, f)) = (P \otimes_R Q, 1 \otimes f)$, by using the categorical definition of K_1 given in [3, p. 348].

When $A = 1$, the pairings T_0 and T_1 reduce to the standard module structure of $G_R(B)$ on $K_0 R(B)$ and $K_1 R(B)$ respectively (cf. [3, p. 565]).

Let P be free over $R(A)$ of finite rank and $\rho: B \to \mathrm{Aut}_R P$ be α-semilinear. By picking a basis $e_1, \ldots, e_m$ for P and setting $\rho(b)(e_i) = \sum_j \beta_{ij} e_j$, we obtain for each $b \in B$ a nonsingular matrix, $(\beta_{ij}) \in \mathrm{GL}(m; R(A))$, whose transpose is called the *matrix of $\rho(b)$ with respect to $e_1, \ldots, e_m$*. ρ is *simple* if for all $b \in B$ the class of $(k(\beta_{ij}))^t$ is zero in $\mathrm{Wh}\, R(A \times_\alpha B)$, where $k: R(A) \to R(A \times_\alpha B)$. This is independent of the choice of basis. Let $S_R(B, A, \alpha) \subset G_R(B, A, \alpha)$ be the subgroup generated by the simple representations. Extending Theorem 1, we have

LEMMA 2. *T_1 induces a pairing $T_1': S_R(B, A, \alpha) \otimes \mathrm{Wh}\, R(B) \to \mathrm{Wh}\, R(A \times_\alpha B)$.*

PROOF. For any $b \in B$, let $f_b: R(B) \to R(B)$ be the $R(B)$ linear map that sends 1 to b. By giving the pairing T_1 a matrix interpretation as in the proof of [**1**, Propositon 1.1], one verifies that if $\rho: B \to \mathrm{Aut}_R P$ is simple, then $T_1([P, \rho] \otimes [R(B), f_b]) = [P \otimes_R R(B), 1 \otimes f_b] = 0$ in $\mathrm{Wh}\, R(A \times_\alpha B)$. Since $\mathrm{Wh}\, R(B) = K_1 R(B)/(\text{subgroup generated by } \pm[R(B), f_b])$ where $b \in B$, the lemma follows.

The pairings T_0, T_1, and T'_1 are natural in a sense we now make precise. Let C be a group and $\gamma: B \to \mathrm{Aut}\, C$. A homomorphism $\sigma: A \to C$ is *admissible* if $\sigma(\alpha(b)(a)) = \gamma(b)\sigma(a)$ for all $a \in A, b \in B$. If $\sigma: A \to C$ is admissible, then $\sigma \times 1: A \times_\alpha B \to C \times_\gamma B$ is a homomorphism. An admissible homomorphism $\sigma: A \to C$ induces a homomorphism $\sigma_*: G_R(B, A, \alpha) \to G_R(B, C, \gamma)$ by setting $\sigma_*[P, \rho] = [R(C) \otimes_{R(A)} P, \gamma \otimes \rho]$ where $R(C)$ becomes a right $R(A)$ module via σ.

THEOREM 3. *Let $\sigma: A \to C$ be admissible. Then*

$$(\sigma \times 1)_* T_i = T_i(\sigma_* \times 1) \quad \textit{for } i = 0, 1$$

and

$$(\sigma \times 1)_* T'_1 = T'_1(\sigma_* \times 1).$$

COROLLARY 4. *The usual module structure $T: G_R(B) \otimes K_i R(B) \to K_i R(B)$ is a direct summand of $T_i: G_R(B, A, \alpha) \otimes K_i R(B) \to K_i R(A \times_\alpha B), i = 0, 1$.*

PROOF. Apply Theorem 3 to the admissible homomorphisms $\sigma: A \to 1$ and $\tau: 1 \to A$ where 1 is the trivial group.

2. **The generalized product theorems.** Let C_* be a finitely generated chain complex of projectives over $R(A)$ and $\rho: B \to \mathrm{Aut}_R(C_*)$ be an α-semilinear representation such that for each $b \in B, \rho(b)$ is a chain map. Let $\rho_i: B \to \mathrm{Aut}_R(C_i)$ denote the restriction of ρ to C_i. We set

$$\chi(C_*, \rho) = \sum (-1)^i [C_i, \rho_i] \in G_R(B, A, \alpha)$$

and call $\chi(C_*, \rho)$ the *Euler characteristic* of ρ.

If D_* is a chain complex over $R(B)$, the proof of Theorem 1 shows that the usual tensor product of chain complexes $C_* \otimes_R D_*$ is a chain complex over $R(A \times_\alpha B)$.

THEOREM 5. *Let C_*, ρ, and D_* be as above and suppose that D_* is dominated by a finitely generated chain complex of projectives over $R(B)$. Then $C_* \otimes_R D_*$ is dominated by a finitely generated chain complex of projectives over $R(A \times_\alpha B)$ and*

$$w(C_* \otimes_R D_*) = \chi(C_*, \rho) w(D_*)$$

where w denotes the Wall invariant and juxtaposition denotes the pairing T_0.

THEOREM 6. *Let C_* be a finitely generated, based chain complex over $R(A)$ and let $\rho : B \to Aut_R(C_*)$ be an α-semilinear representation such that for each i, ρ_i is simple. Let D_* be a finitely generated, based, acyclic chain complex over $R(B)$. Then $C_* \otimes_R D_*$ is a finitely generated, based, acyclic chain complex over $R(A \times_\alpha B)$ and*

$$\tau_R(C_* \otimes_R D_*) = \chi(C_*, \rho)\tau_R(D_*)$$

where τ_R denotes R-Whitehead torsion and juxtaposition denotes the pairing T'_1.

These theorems are obtained by an induction argument similar to that of [**6**, p. 188].

Since $\rho(b)$ is a chain map for each $b \in B$, there are induced α-semilinear representations $\bar{\rho}_i : B \to \mathrm{Aut}_R(H_i(C_*))$. If $H_i(C_*) \in \mathfrak{P}R(A)$ for all i, we set $\chi(H_*(C_*), \bar{\rho}) = \sum(-1)^i[H_i(C_*), \bar{\rho}_i] \in G_R(B, A, \alpha)$.

LEMMA 7. *Let C_* be a finitely generated chain complex of projectives over $R(A)$ such that $H_i(C_*) \in \mathfrak{P}R(A)$ for all i and let $\rho : B \to \mathrm{Aut}_R(C_*)$ be an α-semilinear representation as above. Then $\chi(C_*, \rho) = \chi(H_*(C_*), \bar{\rho})$.*

3. **Flat bundles.** Let $\xi = (E, p, B, F)$ be a PL fiber bundle (cf. [**1**]) and $\pi = \pi_1(B, b_0)$. ξ is *flat* if there is a triangulation K of F with vertex v, and a homomorphism $\omega : \pi \to \mathrm{Iso}(K, v)$, the simplicial isomorphism of K leaving v fixed, such that $\xi = (\tilde{B} \times_\pi F, p', B, F)$ where $\tilde{B}$ is the universal cover of B, $\beta \in \pi$ acts on $\tilde{B} \times F$ by $\beta(x, y) = (\beta x, \omega(\beta)y)$, and $p' : \tilde{B} \times_\pi F \to B$ is the natural map. ξ is called the *flat bundle associated with* $\omega : \pi \to \mathrm{Iso}(K, v)$.

Let $s : B \to E$ be the natural cross section $s(b) = v_b$ where v_b is the image of v in $p^{-1}(b)$ and let $e_0 = s(b_0)$. Then s_* splits the homotopy exact sequence of ξ. Furthermore

LEMMA 8. *Define $\alpha : \pi_1(B, b_0) \to \mathrm{Aut}\, \pi_1(F, v)$ by $\alpha(\beta) = \omega(\beta)_* : \pi_1(F, v) \to \pi_1(F, v)$. Then $i_* \times s_* : \pi_1(F, v) \times_\alpha \pi_1(B, b_0) \to \pi_1(E, e_0)$ is an isomorphism.*

In the remainder of this section we identify these two groups via this isomorphism.

Let ξ be the flat bundle associated with $\omega : \pi \to \mathrm{Iso}(K, v)$. Let $\tilde{K}$ be the triangulation of $\tilde{F}$, the universal cover of F, covering K and $C_*(\tilde{K}; R)$ be the cellular chains on $\tilde{K}$ with R coefficients viewed as a (based) $R\pi_1(F, v)$ module (when necessary). Since each $f \in \mathrm{Iso}(K, v)$ fixes v, f is covered by a simplicial isomorphism $\tilde{f} : (\tilde{K}, \tilde{v}) \to (\tilde{K}, \tilde{v})$ where $\tilde{v}$ is a fixed vertex of $\tilde{K}$ over v. The correspondence $f \to \tilde{f}$ defines a homomorphism $\lambda : \mathrm{Iso}(K, v) \to \mathrm{Iso}(\tilde{K}, \tilde{v})$. Define $\rho : \pi_1(B, b_0) \to \mathrm{Aut}_R(C_*(\tilde{K}; R))$ by

$$\rho(\beta) = \lambda\omega(\beta)_* : C_*(\tilde{K}; R) \to C_*(\tilde{K}; R).$$

Relative to the α of Lemma 8, ρ is α-semilinear.

THEOREM 9. *Let $\xi = (E, p, B, F)$ be the flat bundle associated with $\omega : \pi_1(B, b_0) \to \mathrm{Iso}(K, v)$ and suppose B is dominated by a finite complex and F is compact. Then E is dominated by a finite complex and*

$$w(E) = \chi(C_*(\tilde{K}; Z), \rho)w(B)$$

where w denotes the (unreduced) Wall invariant.

THEOREM 10. *Let $\xi = (E, p, B, F)$ be the flat bundle associated with $\omega : \pi_1(B, b_0) \to \mathrm{Iso}(K, v)$ and suppose that the subcomplex A is a deformation retract of B. Then $E_A = p^{-1}(A)$ is a deformation retract of E and*

$$\tau_R(E, E_A) = \chi(C_*(\tilde{K}; R), \rho)\tau_R(B, A)$$

where τ_R denotes Whitehead torsion over R.

The idea of the proof of these theorems is to note that the universal cover $\tilde{E}$ of E is $\tilde{F} \times \tilde{B}$, that $(\beta, \gamma) \in \pi_1(F, v) \times_\alpha \pi_1(B, b_0)$ acts by $(\beta, \gamma)(x, y) = (\beta\lambda\omega(\gamma)x, \gamma y)$, and, therefore, that $C_*(\tilde{E}; R) \approx C_*(\tilde{F}; R) \otimes_R C_*(\tilde{B}; R)$ as modules over $\pi_1(E, e_0) = \pi_1(F, v) \times_\alpha \pi_1(B, b_0)$. The proof is completed by applying Theorems 5 and 6, respectively.

Suppose now that $\pi_1(F, v)$ is finite. Then $\tilde{K}$ is a finite complex and the $Q\pi_1(F, v)$ module $H_*(\tilde{K}; Q)$ is finitely generated over Q. By Maschke's Theorem [3, p. 559], $H_*(\tilde{K}; Q)$ and $H_i(\tilde{K}; Q)$ are finitely generated projectives over $Q\pi_1(F, v)$. Hence $\chi(C_*(\tilde{K}; Q), \bar{\rho}) = \chi(H_*(\tilde{K}; Q), \bar{\rho})$ by Lemma 7. Since the representation $\bar{\rho} : \pi_1(B, b_0) \to \mathrm{Aut}_Q(H_*(\tilde{K}; Q))$ depends only on the structure ξ as a bundle with structure group $\mathrm{Homeo}(F, v)$ (i.e., ρ depends only on topological properties of the bundle ξ), we have the

COROLLARY 11. *Let $\xi = (E, p, B, F)$ and $A \subset B$ be as in Theorem 10. If $\pi_1(F, v)$ is finite, then the rational Whitehead torsion, $\tau_Q(E, E_A)$, depends only on $\tau(B, A)$ and the topological structure of the bundle ξ.*

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