BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

FINITE HILBERT TRANSFORMS IN L^p

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Communicated by P. D. Lax, November 11, 1971

We provide here the setting for an essentially complete analysis of finite Hilbert transforms as extended to be bounded operators in L^p spaces. Here L^p, $1 < p < \infty$, denotes the Banach space of pth power summable functions with respect to Lebesgue measure over a finite interval of the reals.

The spectrum is described completely and explicitly in all cases and we present the elementary relation that exists between the resolvent and its inverse. The latter, see (9) or (7), (perhaps more appropriately a particular case) could be considered as the analogue of Hilbert's reciprocity formulas for the transform on the whole line. Where point spectrum exists, the eigenfunctions are given and in the case of residual spectrum, the appropriate range is presented simply.

Spectral properties of finite Hilbert transforms in L^p have been described before; in particular, see Widom [**12**], Shamir [**10**], and in case $p = 2$ Koppelman and Pincus [**4**]. The analysis in these is more sophisticated than that which follows. Here the analysis is based mainly on certain formal manipulations in an algebra of elementary operations and proceeds largely from the generalized reciprocity formulas mentioned above.

Finite Hilbert transforms are classical and occur often in many and varied forms, particularly in applications. There are numerous publications where the question of inversion of the transformation in various spaces of functions is considered. We mention only the comprehensive work in Tricomi [**11**] and, in addition, direct attention to the book by Butzer and Trebels [**1**]. The results presented here have relevance as well to many of the studies where Cauchy type singular integrals and/or generalized Abel type integrals enter; see Peters [**8**], Samko [**9**], von Wolfersdorf [**13**], and [**1**]. Here, as above, we have given only certain selected references.

We shall restrict our attention to the interval [0, 1]. On complex valued functions we consider the operations J^β (Riemann-Liouville integral) and $J^{*\beta}$ defined by

AMS 1969 *subject classifications*. Primary 4730, 4635, 4515; Secondary 4450, 2643.
Key words and phrases. Finite Hilbert transform, L^p space, spectral properties.
[1] Partially supported during the course of this work under National Science Foundation Grant GP-21337.

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\begin{aligned}
J^{\beta}\varphi(t) &= \Gamma(\beta)^{-1}\int_0^t (t-x)^{\beta-1}\varphi(x)\,dx, && 0<\operatorname{Re}\beta,\\
(1)\qquad &= \lim_{b\to 0+} J^{b+\beta}\varphi(t)\quad (L^p\text{-limit}), && \operatorname{Re}\beta = 0,\\
&= \frac{d}{dt}J^{\beta+1}\varphi(t), && -1<\operatorname{Re}\beta<0,
\end{aligned}
$$

and

$$
\begin{aligned}
J^{*\beta}\varphi(t) &= \Gamma(\beta)^{-1}\int_t^1 (x-t)^{\beta-1}\varphi(x)\,dx, && 0<\operatorname{Re}\beta,\\
(2)\qquad &= \lim_{b\to 0+} J^{*b+\beta}\varphi(t)\quad (L^p\text{-limit}), && \operatorname{Re}\beta = 0,\\
&= -\frac{d}{dt}J^{*\beta+1}\varphi(t), && -1<\operatorname{Re}\beta<0.
\end{aligned}
$$

For the cases where $\operatorname{Re}\beta = 0$ see, for example, Kalisch [3]. In addition let M^{γ} denote the operation given by

$$(3)\qquad M^{\gamma}\varphi(t) = t^{\gamma}\varphi(t), \qquad \gamma\text{-complex},$$

and R the operation

$$(4)\qquad R\varphi(t) = \varphi(1-t).$$

Let H denote the finite Hilbert transform given by

$$(5)\qquad H\varphi(t) = \frac{1}{\pi}(\text{p.v.})\int_0^1 \frac{\varphi(x)}{t-x}\,dx,$$

the integral being the Cauchy principal value. We also consider H as extended to a bounded operator in L^p $(1<p<\infty)$.

In the algebra of elementary operations generated by the collection $\{J^{\beta}, R, M^{\gamma}\}$ there is the following statement that is basic.

THEOREM 1. *For* $-1<\operatorname{Re}\alpha<1$,

$$(6)\qquad (\cos\pi\alpha)I + (\sin\pi\alpha)H = M^{-\alpha}J^{\alpha}J^{*-\alpha}M^{\alpha}.$$

The equality (6) for $\operatorname{Re}\alpha \neq 0$ is understood as being an arithmetical identity on applying each side to a smooth function and in the sense of L^p $(1<p<\infty)$ when $\operatorname{Re}\alpha = 0$.

Theorem 1 was proved by the author in [2] for α real, $0 \leqq \alpha < 1$. The proof presented there is valid for $0<\operatorname{Re}\alpha<1$ and the result extends by duality to $-1<\operatorname{Re}\alpha<0$. For $\operatorname{Re}\alpha = 0$, (6) is established through a limiting process, as in the definitions (1) and (2), and using results as in Kalisch [3] and Love [5].

THEOREM 2. *For $z = \cot \pi\alpha$, $0 < \operatorname{Re}\alpha < 1$,*

$$(M^{\alpha}RM^{-\alpha})(zI - H)(M^{\alpha}RM^{-\alpha})(zI - H) = (\csc \pi\alpha)^2 I. \tag{7}$$

The equality (7) is valid (in particular) in the sense of providing an arithmetic identity on applying each side to an arbitrary smooth function.

THEOREM 3. *For $|\operatorname{Re}\alpha| < \frac{1}{2}$ the operation*

$$\varphi \to (M^{\alpha}RM^{-\alpha})H(M^{\alpha}RM^{-\alpha})\varphi$$

defines a bounded operator in L^p where $(1 - |\operatorname{Re}\alpha|)^{-1} < p < |\operatorname{Re}\alpha|^{-1}$.

When $z = \cot \pi\alpha$, $0 < \operatorname{Re}\alpha < 1$, it follows from (6) through routine calculations that $(zI - H)f = g$, $g \in C_0^\infty(0, 1)$, where

$$\begin{aligned} f(t) = {} & \frac{(\sin \pi\alpha)^2}{\pi} \frac{t^{\alpha}}{(1 - t)^{\alpha}} \int_0^1 (1 - x)^{\alpha - 1} x^{-\alpha} g(x)\, dx \\ & - \frac{(\sin \pi\alpha)^2}{\pi} t^{\alpha} \int_t^1 (y - t)^{-\alpha} \left(\int_0^y (y - x)^{\alpha - 1} \frac{d}{dx}(x^{-\alpha} g(x))\, dx \right) dy. \end{aligned} \tag{8}$$

Formulas equivalent to (8) for α real appear in [8], Muskhelishvili [7], and Mikhlin [6], for example. It is clear from (8) that $f \in L^p$ if and only if either $p \operatorname{Re}\alpha < 1$ or the first term vanishes. From these observations and the foregoing theorems we are led to the following.

THEOREM 4. (i) *Spectrum* $\operatorname{sp}(H|L^p) = \{\cot \pi\alpha : |\operatorname{Re}\alpha - \frac{1}{2}| \leqq |1/p - \frac{1}{2}|\}$.

(ii) *In case $1 < p < 2$ each z in the interior of* $\operatorname{sp}(H|L^p)$ *is an eigenvalue of multiplicity one and $zI - H$ is onto. For $z = \cot \pi\alpha$, $|\operatorname{Re}\alpha - \frac{1}{2}| < 1/p - \frac{1}{2}$, the eigenfunctions are multiples of $\varphi_z(t) = (1 - t)^{-\alpha} t^{\alpha - 1}$. For $2 < p < \infty$ each point z of the interior is in the residual spectrum and the range of $zI - H$ has deficiency one. The boundary in all cases $1 < p < \infty$ consists of points in the continuous spectrum. For $z = \cot \pi\alpha$, $|\operatorname{Re}\alpha - \frac{1}{2}| = |1/p - \frac{1}{2}|$, the function $\varphi_z(t) = (1 - t)^{-\alpha} t^{\alpha - 1}$ is a generalized eigenfunction.*

(iii) *The resolvent set $\rho(H|L^p) = \{\cot \pi\alpha : |\operatorname{Re}\alpha| < \frac{1}{2} - |1/p - \frac{1}{2}|, \alpha \neq 0\}$ and for $z \in \rho(H|L^p)$ the resolvent*

$$(zI - H)^{-1} = (\sin \pi\alpha)^2 (M^{\alpha}RM^{-\alpha})(zI - H)(M^{\alpha}RM^{-\alpha}) \tag{9}$$

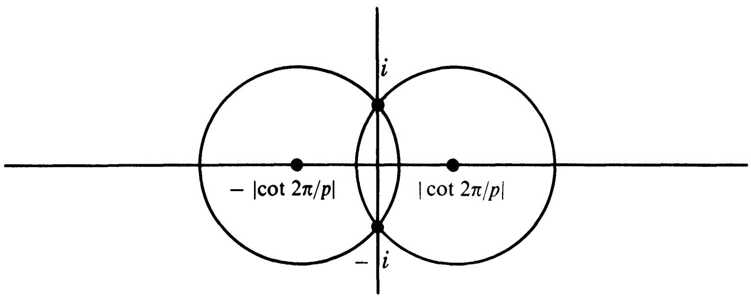
where $z = \cot \pi\alpha$.

(iv) *For $z \in \{\cot \pi\alpha : |\operatorname{Re}\alpha - \frac{1}{2}| < \frac{1}{2} - 1/p\}$, $(zI - H)^{-1}$ maps $(zI - H)(L^p)$ boundedly onto L^p $(2 < p < \infty)$. The subspace $(zI - H)(L^p)$ is the null space of the functional given by $f(t) = (1 - t)^{\alpha - 1} t^{-\alpha}$, $z = -\cot \pi\alpha$.*

COROLLARY. *For $1 < p < \infty$*

$$\|H\|_{L^p} \geqq \max(1, \tan \pi|1/p - \tfrac{1}{2}|). \tag{10}$$

The spectrum has a simple geometrical description. It can be conveniently described and illustrated using the pair of disks bounded by the circles centered at $\pm|\cot 2\pi/p|$ and which pass through $\pm i$.



(a) For $0 < |1/p - \frac{1}{2}| \leqq \frac{1}{4}$ the $\mathrm{sp}(H|L^p)$ is the intersection of the disks.
(b) For $p = 2$ the $\mathrm{sp}(H|L^2)$ is the interval $[-i, i]$.
(c) For $|1/p - \frac{1}{2}| \geqq \frac{1}{4}$ the $\mathrm{sp}(H|L^p)$ is the union of the disks.

References

1. P. L. Butzer and W. Trebels, *Hilberttransformation, gebrochene Integration und Differentiation*, Westdeutscher Verlag, Cologne, 1968. MR **38** #4926.
2. R. K. Juberg, *On the boundedness of certain singular integral operators*, Proc. Conference on Operator Theory, Szeged, Hungary, September, 1970 (to appear).
3. G. K. Kalisch, *On fractional integrals of pure imaginary order in L_p*, Proc. Amer. Math. Soc. **18** (1967), 136–139. MR **35** #7145.
4. W. Koppelman and J. D. Pincus, *Spectral representations for finite Hilbert transformations*, Math. Z. **71** (1959), 399–407. MR **21** #5869.
5. E. R. Love, *Fractional derivatives of imaginary order*, J. London Math. Soc. (2) **3** (1971), 241–259.
6. S. G. Mikhlin, *Integral equations and their applications to some problems of mechanics, mathematical physics and engineering*, GITTL, Moscow, 1949; English transl., Pergamon Press, New York, 1957. MR **12**, 712; MR **19**, 428.
7. N. I. Muskhelišvili, *Singular integral equations. Boundary problems of function theory and their application to mathematical physics*, OGIZ, Moscow, 1946; English transl., Noordhoff, Groningen, 1953. MR **8**, 586; MR **15**, 434.
8. A. S. Peters, *Abel's equation and the Cauchy integral equation of the second kind*, Comm. Pure Appl. Math. **21** (1968), 51–65. MR **36** #6886.
9. S. G. Samko, a) *A generalized Abel equation and fractional integration operators*, Differencial'nye Uravnenija **4** (1968), 298–314. (Russian) MR **37** #3295.
b) *Noether's theory for the generalized Abel integral equation*, ibid. **4** (1968), 315–326. (Russian) MR **37** #3296.
10. E. Shamir, *Reduced Hilbert transforms and singular integral equations*, J. Analyse Math. **12** (1964), 277–305. MR **29** #2617.
11. F. G. Tricomi, *On the finite Hilbert transformation*, Quart. J. Math. Oxford Ser. (2) **2** (1951), 199–211. MR **13**, 231.
12. H. Widom, *Singular integral equations in L^p*, Trans. Amer. Math. Soc. **97** (1960), 131–160. MR **22** #9830.
13. L. von Wolfersdorf, *Zur Lösung der verallgemeinerten Abelschen Integralgleichung mit konstanten Koeffizienten*, Z. Angew. Math. Mech. **49** (1969), 759–761. MR **41** #763.

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