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CLASSIFICATION OF THE COMPLETELY PRIMARY TOTALLY RAMIFIED ORDERS WITH A FINITE NUMBER OF NONISOMORPHIC INDECOMPOSABLE LATTICES

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Let K be the $\mathfrak{p}$-adic completion of an algebraic number field and denote by R its ring of integers. We assume that Λ is an R-order in the semisimple finite dimensional K-algebra A. One of the main problems in the representation theory of orders is the classification of those orders Λ which have only a finite number of nonisomorphic indecomposable left Λ-lattices the so-called "orders of finite lattice type." The commutative case has been settled independently by Drozd-Roiter [2] and Jacobinski [3]. For the general case only partial results are known [1], [4], [5], [7].

By Morita equivalence we may assume that $\Lambda/J(\Lambda)$, where $J(\Lambda)$ denotes the Jacobson-radical of Λ, is a finite direct sum of extension fields $\mathfrak{K}_i$ of $\mathfrak{K}$, the residue field of R, say

$$\Lambda/J(\Lambda) \overset{\text{ring}}{\cong} \bigoplus_{i=1}^{m} \mathfrak{K}_i.$$

We choose a finite unramified extension K' of K with ring of integers R' such that the residue field $\mathfrak{K}'$ of R' is a splitting field for the minimum polynomial of $\mathfrak{K}_i$ over $\mathfrak{K}$. Putting $\Lambda' = R' \otimes_R \Lambda$ we have

$$\Lambda'/J(\Lambda') \cong \bigoplus_{i=1}^{n} \mathfrak{K}'.$$

Jacobinski [3, Proposition 1] has shown that Λ is of finite lattice type if and only if Λ' is of finite lattice type. Therefore we may assume that

$$\Lambda/J(\Lambda) \cong \bigoplus_{i=1}^{n} \mathfrak{K}, \tag{1}$$

where $\mathfrak{K}$ is the residue field of R.

We shall classify here those orders of finite lattice type for which $n = 1$ —these are called "completely primary totally ramified" (notation CPTR). By $n(\Lambda)$ we denote the number of nonisomorphic indecomposable left Λ-lattices.

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THEOREM. *Let* Λ *be a CPTR-order in A.* $n(\Lambda) < \infty$ *if and only if either*

(i) *For every CPTR-overorder* Ω *of* Λ *we have*

(α) *The left ring of multipliers* Ω' *of* $J(\Omega)$ *coincides with the right ring of multipliers of* $J(\Lambda)$.

Put $\mathfrak{A} = \Omega'/J(\Omega)$,

(β) $\dim_{\mathfrak{K}}(\mathfrak{A}) \leqq 3$,

(γ) $\dim_{\mathfrak{K}}(J(\mathfrak{A})/J^2(\mathfrak{A})) \leqq 1$, *where* $J(\mathfrak{A})$ *denotes the Jacobson-radical of* $\mathfrak{A}$,

or

(ii) Λ *is conjugate to*

$$\Omega_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \pi' a_{21} & a_{11} + \pi' a_{22} & a_{12} + \pi' a_{23} \\ \pi' a_{31} & \pi' a_{32} & a_{11} + \pi' a_{33} \end{pmatrix} \middle| \, a_{ij} \in R', 1 \leqq i, j \leqq 3 \right\},$$

where R' *is a totally ramified finite extension of* R *and* $\pi' R' = J(R')$.

REMARK. Ω_1 is the only type of CPTR-order with a finite number of non-isomorphic indecomposable lattices for which the left ring of multipliers of the radical is different from the right ring of multipliers of the radical.

SOME COMMENTS TO THE PROOF. Let

$$A = \bigoplus_{i=1}^{s} (D_i)_{s_i},$$

where D_i are skewfields over K, and assume that Λ is a CPTR-order in A with $n(\Lambda) < \infty$. Then it is shown in [**6**] and [**7**] that

$$\sum_{i=1}^{s} s_i \leqq 3.$$

One shows quite easily that for $s_i = 1$, $1 \leqq i \leqq s$, the conditions of our theorem coincide with the conditions of Drozd-Roiter [**2**] (cf. also [**5**], [**7**]).

In case $s = 1$ and $s_1 = 3$ one shows that $n(\Omega_1) < \infty$ using some results of Kirichenko [**4**], and with [**7**] this case is settled. For $s = 1$ and $s_1 = 2$, one shows that the conditions of our theorem imply that Λ is a Bass-order, and with [**1**], this case is settled. Finally for $s = 2$, $s_1 = 1$ and $s_2 = 2$, one has to do some computations (similar to those in [**2**] and [**5**]) to conclude that our conditions are sufficient for $n(\Lambda) < \infty$.

REMARK. Let Λ be any order satisfying (1). Let $1 = \sum e_i$ be the decomposition of $1 \in \Lambda$ into primitive orthogonal idempotents. Then $\Omega_i = e_i \Lambda e_i$ is a CPTR-order and $n(\Lambda) < \infty$ implies $n(\Omega_i) < \infty$ and so Ω_i is known.

References

1. Ju. A. Drozd and V. V. Kiričenko, *On representations of rings lying in matrix algebras of the second kind*, Ukrain. Mat. Ž. **19** (1967), no. 3, 107–112. (Russian) MR **35** #1632.

2. Ju. A. Drozd and A. V. Roĭter, *Commutative rings with a finite number of indecomposable integral representations*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 783–798 = Math. USSR Izv. **1** (1967), 757–772. MR **36** #3768.

3. H. Jacobinski, *Sur les ordres commutatifs avec un nombre fini de réseaux indécomposables*, Acta Math. **118** (1967), 1–31. MR **35** #2876.

4. V. V. Kiričenko, *Representations of matrix rings of third order*, Mat. Zametki **8** (1970), 235–244. (Russian)

5. K. W. Roggenkamp, *Charakterisierung von Ordnungen in einer direkten Summe kompletter Schiefkörper, die nur endlich viele nicht isomorphe unzerfällbare Darstellungen haben*, Mitt. Math. Sem. Giessen **89** (1971), 1–122.

6. ———, *Some orders of infinite lattice type*, Bull. Amer. Math. Soc. **77** (1971), 1055–1056.

7. ———, *Some necessary conditions for orders to be of finite lattice type*. I, II, J. Reine Angew. Math. (to appear).

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