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WEAK COMPACTNESS IN THE SPACE OF VECTOR MEASURES[1]

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ABSTRACT. A necessary and sufficient condition for weak compactness in the space of vector measures is given.

1. **Introduction and statement of the theorem.** Weak compactness in the space of set functions has been studied by a number of authors. Dubrovskiĭ [**6**], Grothendieck [**9**], Bartle, Dunford and Schwartz [**1**] obtained necessary and sufficient conditions for weak compactness in the space of countably additive scalar measures. Leader [**10**] and Porcelli [**13**] treated the finitely additive scalar case. Chatterji [**4**] considered weak compactness in $L_1(\mathfrak{X})$; a slightly more general case was studied by Batt and Berg [**2**].

In this paper we give necessary and sufficient conditions for weak compactness in a very general setting, which includes the above results as special cases. Details and applications will be presented elsewhere. Our theorem is the following (definitions are given in the next section).

THEOREM. *Let $\mathfrak{X}$ be a reflexive Banach space and let $\mathscr{R}$ be a ring of sets. A set $K \subset fa(\mathscr{R}, \mathfrak{X})$ is conditionally weakly compact if and only if K is bounded and the set of measures $\{v(\mu):\mu \in K\}$ is uniformly additive.*

REMARK 1. In the case K is weakly compact, there exists a positive bounded finitely additive set function λ such that $\{v(\mu):\mu \in K\}$ is uniformly absolutely continuous with respect to λ. Conversely, if K is bounded and such a λ exists, then K is conditionally weakly compact. The hypothesis of reflexivity in the necessary part of Theorem 1 can be omitted.

COROLLARY 1. *The above theorem remains valid if $fa(\mathscr{R}, \mathfrak{X})$ is replaced by $ca(\mathscr{R}, \mathfrak{X})$.*

REMARK 2. The theorem is false for every $\mathfrak{X}$ which is not reflexive. This observation was made by J. J. Uhl, Jr. In fact, if $\mathfrak{X}$ is not reflexive, choose a sequence $\{x_n\}$ in the unit sphere of $\mathfrak{X}$ which has no weakly convergent subsequences. Assume (S, Σ, μ) is a measure space; let E be a measurable set of positive measure. The subspace $\{x\zeta_E : x \in X\}$ of $L_1(\mu, \mathfrak{X})$ is linearly homeomorphic to $\mathfrak{X}$. Note that the set $K = \{x_n\xi_E : n = 1, 2, \ldots\}$ is not conditionally weakly compact.

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COROLLARY 2. *$fa(\mathscr{R}, X)$ is weakly sequentially complete if X is reflexive.*

2. **Notation and definition.** $\mathfrak{X}$ denotes a Banach space with norm $\| \ \|$ and conjugate space $\mathfrak{X}^*$. Let $\mathscr{R}$ be a ring of subsets of a set S. For a set function $\mu:\mathscr{R} \to \mathfrak{X}$, define the total variation set function on the power set of S by $v(\mu)(A) = \sup \sum \|\mu(R_i)\|$, where the supremum is taken over all finite disjoint families of sets R_i contained in A. $fa(\mathscr{R}, \mathfrak{X})$ is the Banach space of all finitely additive set functions μ defined on $\mathscr{R}$ with $v(\mu)(S) < \infty$. The norm of μ is $v(\mu)(S)$. $ca(\mathscr{R}, \mathfrak{X})$ is the subspace of $fa(\mathscr{R}, \mathfrak{X})$ consisting of countably additive set functions. $\sigma(\Sigma)$ denotes the σ-algebra of sets generated by Σ. $L_1(S, \mathscr{A}, v, \mathfrak{X})$ is the space of Bochner integrable functions relative to the measure space $(S, \mathscr{A}, v)$; $L_\infty(S, \mathscr{A}, v, \mathfrak{X})$ denotes the class of $\mathscr{A}$-measurable essentially bounded $\mathfrak{X}$-valued functions. A family Γ of finitely additive scalar measures of bounded variation defined on $\mathscr{R}$ is *uniformly additive* if for every disjoint sequence of sets R_i, $\lim_n \sum_{i=n}^{\infty} |\mu(R_i)| = 0$ uniformly for $\mu \in \Gamma$. Weak convergence is denoted by $\xrightarrow{w}$.

3. **Brief outline of the proof.** I. Assume $\mathscr{R} = \Sigma$ is a σ-algebra. Let $K = \{\mu_i\}_{i=1}^{\infty} \subset ca(\Sigma, \mathfrak{X})$. Suppose $\mu_0:\Sigma \to \mathfrak{X}$ is a set function such that $\mu_i(E) \xrightarrow{w} \mu_0(E)$ in $\mathfrak{X}$ for every $E \in \Sigma$. Then for every $x^* \in \mathfrak{X}^*$, $x^*\mu_0 = \lim x^*\mu_i$. By the Nikodym theorem [7, p. 160], $x^*\mu_0$ is countably additive, and the Orlicz-Pettis theorem [11] implies that μ_0 is countably additive on Σ. The expression $\sum_{i=1}^{n} \|\mu_0(E_i)\| = \lim_k \sum_{i=1}^{n} x_i^*\mu_k(E_i)$ for appropriate $x_i^* \in \mathfrak{X}^*$, $\|x_i^*\| \leqq 1$, shows that $v(\mu_0)(S) \leqq \underline{\lim}\, v(\mu_i)(S)$; hence $\mu_0 \in ca(\Sigma, \mathfrak{X})$. Let $\mathfrak{Z}_0$ be the rational span of $\{\mu_i\}_{i=0}^{\infty}$. Obtain a countable algebra $\mathscr{A}_0 \subset \Sigma$ such that $v(\beta/\mathscr{A}_0)(S) = V(\beta)(S)$, $\beta \in \mathfrak{Z}_0$. We then show that $\beta \in \mathfrak{Z}$ implies:

$$(*) \qquad v(\beta/\mathscr{A})(S) = v(\beta)(S),$$

where $\mathfrak{Z}$ is the closure in $ca(\Sigma, \mathfrak{X})$ of $\mathfrak{Z}_0$ and $\mathscr{A} = \sigma(\mathscr{A}_0)$. Let the separable space $\mathfrak{X}_0$ be the closure of $\operatorname{span}\{\beta(A):A \in \mathscr{A}_0, \beta \in \mathfrak{Z}_0\}$. Using the Hahn-Banach theorem and the Hahn extension theorem for measures, we obtain: $\beta(A) \in \mathfrak{X}_0$ whenever $\beta \in \mathfrak{Z}$ and $A \in \mathscr{A}$. This fact and (*) yield the isometry $\pi:\mathfrak{Z} \to ca(\mathscr{A}, \mathfrak{X}_0)$, where $\pi(\beta) = \beta/\mathscr{A}$, $\beta \in \mathfrak{Z}$. Let $v = \sum_{i=0}^{\infty} 2^{-i} v(\mu_i)$. Since $\mathfrak{X}_0$ is reflexive, by the Radon-Nikodym-Phillips theorem [12], we obtain functions f_i in $L_1(S, \mathscr{A}, v, \mathfrak{X}_0) \equiv L_1(\mathfrak{X}_0)$ such that $f_i = d\mu_i/dv$, $i = 0, 1, \ldots$. Since $\mathfrak{X}_0$ is reflexive, $\mathfrak{X}_0^*$ is separable, and thus $L_1(\mathfrak{X}_0)^* = L_\infty(S, \mathscr{A}, v, \mathfrak{X}_0^*) \equiv L_\infty(\mathfrak{X}_0^*)$ (cf. Dinculeanu [5, p. 282]). $\mu_i(E) \xrightarrow{w} \mu_0(E)$ for every E implies that $\int hf_i\, dv \to \int hf_0\, dv$ for simple functions $h \in L_\infty(\mathfrak{X}_0^*)$. Since the $v(\mu_i)$ are uniformly additive and absolutely continuous with respect to v, a result of Gould [8, p. 199] implies that the $v(\mu_i)$ are uniformly absolutely continuous with respect to v. Using this fact together with Egoroff's theorem for $\mathfrak{X}_0^*$-valued measurable functions and the boundedness of $\{v(\mu_i)(S)\}$, one can show that $\int hf_i\, dv \to \int hf_0\, dv$, $h \in L_\infty(\mathfrak{X}_0^*)$; this in turn implies that $\mu_i \xrightarrow{w} \mu_0$ in $ca(\Sigma, \mathfrak{X})$.

II. Suppose $K \subset ca(\Sigma, \mathfrak{X})$, where Σ is a σ-algebra. By the Eberlein-Šmulian theorem, it suffices to consider a sequence $\{\mu_i\} \subseteq K$. As in I, consider $\{\mu_i\} \subset ca(\mathscr{A}, \mathfrak{X}_0)$, $\mathfrak{X}_0$ separable. An application of the Bartle-Dunford-Schwartz weak compactness theorem [**1**] shows that $\{x^*\mu_i\}$ is conditionally weakly compact for every $x^* \in \mathfrak{X}^*$. Let x_n^* be dense in the unit sphere of $\mathfrak{X}_0^*$. By a diagonal process obtain $\{\mu_{i_k}\}$ such that $\lim_k x_n^*\mu_{i_k}(E)$ exists for every $E \in \mathscr{A}$ and every n. The limit also exists then for every $x^* \in \mathfrak{X}_0^*$. Define $\mu(E): \mathfrak{X}_0^* \to$ (scalar field) by $\mu(E)x^* = \lim x^*\mu_{i_k}(E)$. Then $\mu(E) \in \mathfrak{X}_0^{**} = \mathfrak{X}_0$ and $\mu_{i_k}(E) \xrightarrow{w} \mu(E)$, $E \in \mathscr{A}$. As in I, $\mu_{i_k} \xrightarrow{w} \mu$ in $ca(\mathscr{A}, \mathfrak{X}_0)$. Since closed subspaces contain weak limits of elements in the subspace, $\mu_{i_k} \xrightarrow{w} \pi^{-1}(\mu)$ in $ca(\Sigma, \mathfrak{X})$.

III. Let $K \subset fa(\Sigma, \mathfrak{X})$, where Σ is an algebra. Let Σ_1 be the Stone algebra [**7**, p. 312] of all open-closed subsets of the compact Hausdorff space S_1; $\Sigma_2 = \sigma(\Sigma_1)$. For a scalar or vector finitely additive set function μ of bounded variation defined on Σ, let $\tau(\mu)$ be its extension on Σ_2. $\tau: fa(\Sigma, \mathfrak{X}) \to ca(\Sigma_2, \mathfrak{X})$ is an isometry (cf. Uhl [**14**]). It can be proved that on Σ_2, $v(\tau\mu) \leqq \tau(v(\mu))$. The conditional weak compactness of K is established by showing the uniform additivity of $\tau(K)$. This is indicated in the next step.

IV. Let Γ be a bounded set of scalar countably additive set functions defined on an algebra Σ_0. If Γ is a uniformly additive family, then the extensions of Γ to $\sigma(\Sigma_0)$ are also uniformly additive. The technique used to prove this is similar to that found in Brooks [**3**]. A diagonal process is used in addition to results of Leader [**10**] and Porcelli [**13**].

V. Let $\mathscr{R}$ be a ring ($S \notin \mathscr{R}$), Σ the algebra generated by $\mathscr{R}$. Σ consists of sets of the form R or $R \cup (S - R_0)$, $R, R_0 \in \mathscr{R}$ ($R \cap (S - R_0) = \varnothing$). For $\mu \in fa(\mathscr{R}, \mathfrak{X})$, define $\tilde{\mu}$ on Σ by $\tilde{\mu}(R \cup (S - R_0)) = \mu(R) - \mu(R_0)$. Thus $v(\mu)(S) \leqq v(\tilde{\mu})(S) \leqq 2v(\mu)(S)$. K is uniformly additive on $\mathscr{R}$ if and only if K is uniformly additive on Σ. Using the maps $\beta \to \beta/\mathscr{R}$ ($\beta \in fa(\Sigma, \mathfrak{X})$) and $\beta \to \tilde{\beta}$ ($\beta \in fa(\mathscr{R}, \mathfrak{X})$), we reduce V to III.

VI. The converse (without reflexivity) is proved by sliding hump methods in sequence spaces.

VII. Remark 1 follows from the result: *If a family Γ of finitely additive scalar measures is bounded, then Γ is uniformly additive if and only if there exists a bounded positive finitely additive measure λ such that Γ is uniformly absolutely continuous with respect to λ and $\lambda(E) \leqq \sup\{v(\mu)(E): \mu \in \Gamma\}$.* Results from [**1**], [**3**] and [**13**] are used to prove this.

Corollary 1 follows from the theorem by observing that $ca(\mathscr{R}, \mathfrak{X})$ is a closed subspace of $fa(\mathscr{R}, \mathfrak{X})$ and therefore contains weak limits in $fa(\mathscr{R}, \mathfrak{X})$ of elements belonging to $ca(\mathscr{R}, \mathfrak{X})$.

REMARK 3. The author and N. Dinculeanu have recently extended the above theorem to the space of vector measures of local finite variation. A "Synthesis theorem" concerning the existence of a control measure for a family of locally equivalent measures is used to show the existence of a positive measure λ such that weakly compact sets in this locally convex linear topological space are locally uniformly absolutely continuous with respect to it.

REMARK 4. The author has obtained criteria for strong compactness in the space of vector measures with local finite variation. Details will appear elsewhere.

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