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p-ADIC CURVATURE AND A CONJECTURE OF SERRE

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1. In this note we announce a vanishing theorem for the cohomology of discrete subgroups of p-adic groups. The methods and results bear a striking analogy with the real case (see Matsushima [**4**]). In particular, we define "p-adic curvature" for the p-adic symmetric spaces of Bruhat-Tits (see [**1**]). As in the real case, we then reduce the proof of our vanishing theorem to the assertion that the minimum eigenvalues of certain p-adic curvature transformations are sufficiently large. This last condition can then be verified for "sufficiently large" residue class fields.

Before giving a more detailed description of our results we introduce some notation. Thus let $\boldsymbol{Z}$ denote the ring of rational integers and $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ the fields of rational, real and complex numbers, respectively. For a prime p, $\boldsymbol{Q}_p$ will denote the p-adic completion of $\boldsymbol{Q}$. More generally k_v will denote a nondiscrete, totally disconnected, and locally compact (commutative) field.

Let $\boldsymbol{G}$ denote a simply-connected, linear algebraic group defined and simple over k_v and let $\boldsymbol{G}_{k_v}$ denote the k_v-rational points of $\boldsymbol{G}$. Let $V_{\boldsymbol{Q}}$ denote a finite-dimensional vector space over $\boldsymbol{Q}$, Γ an abstract group, and $\rho:\Gamma \to \operatorname{Aut} V_{\boldsymbol{Q}}$ a representation. Let $H^i(\Gamma, \rho)$ denote the ith Eilenberg-Mac Lane group of Γ with respect to ρ. If $V = \boldsymbol{Q}$, and ρ is the trivial representation we write $H^i(\Gamma, \boldsymbol{Q})$ in place of $H^i(\Gamma, \rho)$. By a *uniform lattice* in $\boldsymbol{G}_{k_v}$ we mean a discrete subgroup $\Gamma \subset \boldsymbol{G}_{k_v}$ such that $\boldsymbol{G}_{k_v}/\Gamma$ is compact.

THEOREM 1. *For every integer l, there is an integer $N(l)$ such that if the residue class field of k_v has at least $N(l)$ elements, if $\Gamma \subset \boldsymbol{G}_{k_v}$ is a uniform lattice, and if $\rho:\Gamma \to \operatorname{Aut} V_{\boldsymbol{Q}}$ is a finite-dimensional representation such that $\rho(\Gamma)$ is contained in the orthogonal group of some positive-definite quadratic form on $V_{\boldsymbol{Q}}$, then $H^i(\Gamma, \rho) = 0$ for $0 < i < l$. In particular, $H^i(\Gamma, \boldsymbol{Q}) = 0$ for $0 < i < l$.*

Except for our restriction on the residue class field, this theorem answers a question raised by J.-P. Serre for the trivial representation. It seems likely that sharper estimates for the minimum eigenvalues of p-adic curvature will enable one to eliminate the restriction on the residue class field. For nontrivial ρ, we have had to introduce the additional restriction that $\rho(\Gamma)$ is contained in the orthogonal group of a positive-definite form on $V_{\boldsymbol{Q}}$. Serre has conjectured that if ρ is obtained from a representation of $\boldsymbol{G}_{k_v}$ in

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$V_{\mathbf{Q}} \otimes_{\mathbf{Q}} k_v$, then this additional restriction is unnecessary. Incidentally, Serre observed that $H^i(\Gamma, \rho) = 0$ for $i > l$. For $i = l$ we have the following result (also observed by Borel and Serre):

THEOREM 2. *If Γ is a torsion-free lattice in $\mathbf{G}_{k_v}$, then* $\dim_{\mathbf{Q}} H^l(\Gamma, \mathbf{Q})$ $=$ *multiplicity of the Steinberg representation in* $L^2(\mathbf{G}_{k_v}/\Gamma)$.

(See Shalika [**6**] for the definition of the Steinberg representation in the Chevalley case—one must make the appropriate translation from the adjoint group to the simply-connected group.) Our main theorem (Theorem 1) together with Theorem 2 yields a partial p-adic analogue of the result in Schmid [**5**].

2. **An indication of the proof.** Let $\Gamma \subseteq \mathbf{G}_{k_v}$ be a uniform lattice. For simplicity, assume Γ is torsion-free and ρ is the trivial representation (so $V_{\mathbf{Q}} = \mathbf{Q}$). Let $\mathscr{T}$ be the Bruhat-Tits complex associated with $\mathbf{G}_{k_v}$ (see [**1**]). Then $\mathscr{T}$ is a simplicial complex and is contractible. $\mathbf{G}_{k_v}$ acts on $\mathscr{T}$ as a group of simplicial transformations (and we make the convention that $\mathbf{G}_{k_v}$ acts to the left). The action of Γ is then properly discontinuous and fixed-point free. Hence, it follows that

$$H^i(\Gamma, \mathbf{Q}) \cong H^i(\Gamma\backslash\mathscr{T}, \mathbf{Q}). \tag{2.1}$$

Since $\Gamma\backslash\mathbf{G}_{k_v}$ is compact, one knows that $\Gamma\backslash\mathscr{T}$ is a finite complex which for simplicity, we assume to be a simplicial.

Following Eckmann and Hodge, we describe a "harmonic theory" for the $\mathbf{Q}$-cochains of a finite simplicial complex $\mathscr{S}$. First, for each geometric simplex σ of $\mathscr{S}$, fix an oriented simplex $\hat{\sigma}$ corresponding to σ. By a *Riemannian metric* on $\mathscr{S}$ we will mean a function from the geometric simplices to the positive real numbers. We fix a *Riemannian metric* λ on $\mathscr{S}$, and assume λ is $\mathbf{Q}$-valued. Let f and g be oriented q-cochains on $\mathscr{S}$ with values in $\mathbf{Q}$. We set

$$f, g) = \sum_{\sigma} f(\hat{\sigma})g(\hat{\sigma})\lambda(\sigma),$$

where the sum is over all geometric q-simplices σ of $\mathscr{S}$. We let C^q denote the oriented q-cochains of $\mathscr{S}$ with values in $\mathbf{Q}$. Then (,) defines a positive-definite inner product on C^q. Let $d: C^q \to C^{q+1}$ denote the simplicial coboundary. We let $\delta: C^q \to C^{q-1}$ denote the adjoint of d, with respect to (,) and we set $\Delta = d\delta + \delta d$. Since (,) is positive-definite, it is easy to see that, for $f \in C^q$,

$$\begin{aligned} d\delta f &= 0 \text{ implies } \delta f = 0, \\ \delta d f &= 0 \text{ implies } df = 0, \\ \Delta f &= 0 \text{ implies } df = 0 \text{ and } \delta f = 0. \end{aligned} \tag{2.2}$$

We let $\mathbf{H}^q \subset C^q$ be the kernel of Δ (and we call an element of $\mathbf{H}^q$ a harmonic cochain). From (2.2) it is elementary to prove (see Kostant [**3**]):

PROPOSITION (HODGE DECOMPOSITION). *C^q has (relative to (,)) an orthogonal direct sum decomposition*

$$C^q = \mathbf{H}^q \oplus dC^{q-1} \oplus \delta C^{q+1}.$$

It follows that $\boldsymbol{H}^q \cong H^q(\mathcal{S}, \boldsymbol{Q})$. Thus to prove $H^q(\mathcal{S}, \boldsymbol{Q}) = 0$, it suffices to prove a vanishing theorem for $\boldsymbol{H}^q$. Now for *real* locally symmetric spaces, Matsushima has reduced the study of harmonic forms to the computation of the minimum eigenvalues of certain curvature transformations (see [**4**]). By analogy, we might hope that we can reduce the study of $\boldsymbol{H}^q$ to the computation of the minimum eigenvalue of certain *combinatorial* curvature transformations. Indeed, for $\mathcal{S} = \Gamma\backslash\mathcal{T}$, we can do exactly that for suitably defined curvature transformations. We can then estimate the minimum eigenvalue well enough to prove Theorem 1.

Presently we shall define our curvature transformation for $q < l$ (for an arbitrary finite simplicial complex with Riemannian metric). First we make some preliminary definitions. By the unit sphere Σ about a vertex σ in $\mathcal{S}$, we will mean the boundary of the star of σ. Now for $q > 0$, every q-simplex τ of $\mathcal{S}$, having σ as a vertex, determines a $(q-1)$-simplex τ' of Σ. We set $\lambda'(\tau') = \lambda(\tau)$; then λ' is a Riemannian metric on Σ. We let d' denote the simplicial coboundary on Σ and δ' the adjoint of d' (with respect to λ'). Then for $q < l$ our curvature transformation (at σ) is just $\delta' d'$ acting on the space of $\boldsymbol{Q}$-valued q-cochains of Σ (orthogonal to the constant cochains if $q = 1$).

For example, if $\boldsymbol{G}_{k_v} = SL_3(k_v)$, and F denotes the residue class field of k_v, then every unit sphere Σ in $\Gamma\backslash\mathcal{T}$ is isomorphic as a simplicial complex to Σ_0, which we now define. Let F^3 denote the three-dimensional vector space over F. The set of vertices of Σ_0 will be the set of all lines and planes (containing the origin) in F^3. Two vertices span a one-simplex (Σ_0 is one-dimensional) if and only if one is a line, the other a plane, and the line is contained in the plane. We may take λ' to be identically one. In this case O. Rothaus showed me the eigenvalues of $\delta' d'$. They are 0, 2, $1 \pm q^{1/2}/(q+1)$, where q is the cardinality of F. For general rank 2 groups G, the needed computation is contained in Feit-Higman [**2**]. One can also use the generators and relations of the Hecke algebra. Finally, the general case is handled by an induction argument, starting with the Feit-Higman result.

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