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LIMIT THEOREMS FOR CONTINUOUS STATE BRANCHING PROCESSES WITH IMMIGRATION

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Introduction. K. Kawazu and S. Watanabe [**5**] have defined a CBI *process* as a Markov process $X = (x_t, P_x)$ with state space $[0, \infty]$, with ∞ as a trap, possessing the property that, for each $t \geqq 0, \lambda \geqq 0$, there exist $\varphi(t, \lambda) \geqq 0$ and $\psi(t, \lambda) \geqq 0$ such that

$$E_x[e^{-\lambda x_t}; t < e_\infty] = \varphi(t, \lambda)e^{-x\psi(t,\lambda)}, \tag{1.1}$$

for every $x \in [0, \infty]$; here $e_\infty = \inf\{t : x_t = \infty\}$. Previously Lamperti [**6**] had treated the case $\varphi \equiv 1$.

The Markov property of X implies that, for $\lambda \geqq 0, s, t \geqq 0$,

$$\psi(t + s, \lambda) = \psi(t, \psi(s, \lambda)), \tag{1.2}$$

$$\varphi(t + s, \lambda) = \varphi(t, \lambda)\varphi(s, \psi(t, \lambda)). \tag{1.3}$$

Under the condition of right continuity of X at $t = 0$, it follows from (1.2) and (1.3) that ψ and φ are differentiable. Explicitly, we have

$$\partial\psi/\partial t = R(\psi), \qquad \psi(0^+, \lambda) = \lambda, \tag{1.2'}$$

$$\varphi(t, \lambda) = \exp\left(-\int_0^t F(\psi(s, \lambda))\, ds\right) \tag{1.3'}$$

for appropriate functions R and F. Kawazu and Watanabe have used the property (1.1) to show that they must have the form

$$R(\lambda) = -\alpha\lambda^2 + \beta\lambda + \gamma - \int_{0+}^\infty \left(e^{-\lambda x} - 1 + \frac{\lambda x}{1 + x^2}\right) n_1(dx), \tag{1.4}$$

$$F(\lambda) = c + d\lambda - \int_{0+}^\infty (e^{-\lambda x} - 1) n_2(dx), \tag{1.5}$$

when n_1 and n_2 are measures on the Borel sets of $(0, \infty)$ with the property that

$$\int_{0+}^\infty \frac{u^2}{1 + u^2} n_1(du) + \int_{0+}^\infty \frac{u}{1 + u} n_2(du) < \infty; \qquad \alpha \geqq 0, \gamma \geqq 0, c \geqq 0, d \geqq 0.$$

Furthermore any set of parameters $(\alpha, \beta, \gamma, c, d, n_1, n_2)$ define a unique

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CBI process. In this note we shall only deal with conservative processes. According to Kawazu and Watanabe, this is equivalent to $\gamma = c = 0$ and $\int_{0+}^{\infty} R^*(\lambda)^{-1}\, d\lambda = +\infty$ where $R^*(\lambda) = \max(R(\lambda), 0)$. This is satisfied, for instance, in the case $\int_0^{\infty} x n_1(dx) < \infty$, which we shall explicitly assume; let

$$\rho = R'(0) = \beta - \int_0^{\infty} \frac{x^3}{1 + x^2} n_1(dx).$$

We shall first give a general result and then proceed to examine special cases.

Statement of results.

THEOREM. *Let $X = (x_t, P_x)$ be a conservative CBI process with $\int_1^{\infty} x n_1(dx) < \infty$. Let $\rho(t) = e^{\rho t}$ if $\rho > 0$; $\rho(t) = 1$ if $\rho \leqq 0$. As $t \to \infty$, $x_t/\rho(t)$ converges in distribution to a proper random variable if and only if*

$$\text{(A)} \qquad \int_0^1 \frac{F(\lambda)}{|R(\lambda)|}\, d\lambda < \infty.$$

COROLLARY 1. *Let $\rho > 0$, $\int_1^{\infty} x \log x\, n_1(dx) < \infty$. Then as $t \to \infty$, $x_t/e^{\rho t}$ has a proper, nondegenerate limiting distribution if and only if*

$$\text{(B)} \qquad \int_1^{\infty} (\log x) n_2(dx) < \infty.$$

The convergence takes place almost surely and in L^1 mean.

COROLLARY 2. *Let $\rho < 0$. Then as $t \to \infty$, x_t has a proper, nondegenerate limiting distribution if and only if* (B) *is satisfied.*

COROLLARY 3. *Let $\rho = 0$, $\int_1^{\infty} x^2 n_1(dx) = \infty$. Then as $t \to \infty$, x_t has a proper, nondegenerate limiting distribution if and only if* (A) *is satisfied.*

For comparison with known theorems for Galton-Watson processes we give the following result which applies in the case of finite variance.

COROLLARY 4. *Let $\rho = 0$, $\int_1^{\infty} x^2 n_1(dx) < \infty$, $\int_1^{\infty} x n_2(dx) < \infty$; then as $t \to \infty$, x_t/t has a proper, nondegenerate limiting distribution.*

A short calculation shows that the condition (B) is a special case of the condition (A) for the case $R(\lambda) = \lambda$. This case appeared [**1**] in the study of a storage system proposed by Moran [**7**]. Condition (B) has appeared in the study of discrete parameter, discrete state branching processes, by Heathcote [**3**], [**4**]. Corollary 3 is related to a result of Seneta [**8**]. Recent work of Foster and Williamson [**2**] extend Seneta's observations.

When condition (B) fails, the following result gives a nonlinear normalization which produces weak convergence. We know of no analogue in the discrete parameter case. For simplicity, we state the result in the subcritical case.

THEOREM 2. *Let* $X = (x_t, P_x)$ *be a conservative CBI process with* $-\infty < \rho < 0$. *For* $x > 0$ *let*

$$H(x) = \int_{e^{-x}}^{1} \frac{F(u)}{R(u)}\,du, \quad m(x) = \exp(H(\log x)).$$

Assume that as $x \to \infty$, *we have*

(C1) $$H(x) \to \infty,$$

(C2) $$xH'(x) \to 0.$$

Then for $0 \leqq u \leqq 1$,

(*) $$P_x\{m(x_t)/m(e^{ct}) \leqq u\} \to u^{1/c},$$

as $t \to \infty$, *here* $c = -\rho < 0$.

This result covers cases in which the integral (B) diverges "slowly." Condition (C2) holds, for example, if $H(x) = \log\log x$; then $(\log\log x_t)/(\log ct)$ converges weakly to a limit. If $H(x) = \log x$, condition (C2) fails; a direct calculation shows nonetheless that we have $(\log x_t)/ct$ weakly convergent when $t \to \infty$. If $H(x) = x^{1/2}$, a direct calculation shows that, as $t \to \infty$, $(\log x_t)/t^2$ converges weakly; this is not of the above form (*).

Professor Michael B. Marcus has made the useful observation that $H(x)$ can be expressed directly in terms of the distribution n_2 by the relation $H(x) \sim \text{const} \int_1^x (n_2([u, \infty))/u)\,du)$.

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