BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

LÖWENHEIM-SKOLEM AND INTERPOLATION THEOREMS IN INFINITARY LANGUAGES[1]

BY DAVID W. KUEKER

Communicated by F. W. Gehring, September 15, 1971

Let L be a first-order finitary predicate language with equality. For each pair of infinite cardinals κ and λ with $\kappa \geqq \lambda$ we let $L_{\kappa\lambda}$ be the logic extending L which allows the conjunction ($\wedge$) and disjunction ($\vee$) of fewer than κ formulas and the simultaneous universal or existential quantification of fewer than λ variables. We set $L_{\infty\lambda} = \bigcup_\kappa L_{\kappa\lambda}$. The standard syntactical and semantical concepts are defined as usual (see [**1**], [**2**]). If θ is a sentence we write $\mathfrak{A} \models \theta$ to mean that θ is true on the model $\mathfrak{A}$. $\mathfrak{A} \equiv_{\kappa\lambda} \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ have the same true sentences of $L_{\kappa\lambda}$. $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}_i$ are always used for models for L, and we follow the convention that their universes are A, B, A_i respectively. The cardinality of a set X is denoted by $|X|$. If L' is some other language, then $L'_{\kappa\lambda}$ is the corresponding infinitary logic built on L'. For ease in stating many of our results we assume, except in the last section, that L has only countably many nonlogical symbols. A detailed presentation of these and related results is in preparation for publication elsewhere.

1. $L_{\infty\omega}$ **and the Löwenheim-Skolem theorem.** One form of the downward Löwenheim-Skolem theorem for sentences of $L_{\omega_1\omega}$ can be stated as follows:

(A) If $\mathfrak{A} \models \theta$, then $\mathfrak{A}_0 \models \theta$ for some countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$. The conclusion of (A) is quite weak; certainly the converse does not generally hold. One of our first goals is to define a notion of "almost all" such that the following biconditional holds for sentences of $L_{\omega_1\omega}$:

(B) $\mathfrak{A} \models \theta$ iff $\mathfrak{A}_0 \models \theta$ for almost all countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$. More importantly, we also generalize (B) to apply to sentences of $L_{\infty\omega}$ (for which (A) usually fails). To do this we must first index the countable submodels of a model and define countable approximations to any sentence of $L_{\infty\omega}$.

Let κ be an uncountable cardinal. We define a filter D over $\mathscr{P}_{\omega_1}(\kappa)$, the countable subsets of κ, as follows:

DEFINITION. $X \subseteq \mathscr{P}_{\omega_1}(\kappa)$ belongs to D iff X contains some X' such that (i) for every $s \in \mathscr{P}_{\omega_1}(\kappa)$ there is some $s' \in X'$ such that $s \subseteq s'$ and (ii) X' is closed under unions of countable chains.

LEMMA. *D is a countably complete filter, and if $X_\xi \in D$ for all $\xi < \kappa$ then $\{s : s \in X_\xi$ for all $\xi \in s\} \in D$.*

AMS 1970 *subject classifications*. Primary 02H10, 02B20, 02B25.

[1] Research supported in part by the National Science Foundation under grant NSF GP 20298.

DEFINITION. Let $\mathfrak{A}$ be a model with $|A| \leqq \kappa$. Let $A = \{a_\xi : \xi < \kappa\}$. If $s \in \mathscr{P}_{\omega_1}(\kappa)$ we define $\mathfrak{A}_s$ to be the submodel of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in s\}$.

Terminology. "For almost all s" means "for all s in some set belonging to D." "For almost all countable submodels of $\mathfrak{A}$" means "for $\mathfrak{A}_s$ for almost all s."

REMARKS. (1) $\mathfrak{A}_s$ is almost independent of enumeration of the elements of A; that is, if $A = \{a'_\xi : \xi < \kappa\}$ then $\{a_\xi : \xi \in s\} = \{a'_\xi : \xi \in s\}$ for almost all s. "Almost all countable submodels of $\mathfrak{A}$" therefore has a definite meaning independent of the cardinal $\kappa \geqq |A|$ and the enumeration of A.

(2) The filter D has a game-theoretic characterization. If $X \subseteq \mathscr{P}_{\omega_1}(\kappa)$ we define the game G_x played as follows: I and II alternately choose elements of κ; I wins if the resulting set of their choices belongs to X, and II wins otherwise. Then I has a winning strategy for G_x iff $X \in D$.

For the next definition we assume that the formulas of a conjunction or disjunction in $L_{\kappa^+\omega}$ are indexed by κ.

DEFINITION. Let θ be a formula of $L_{\kappa^+\omega}$. We define its approximations θ^s for $s \in \mathscr{P}_{\omega_1}(\kappa)$ by induction:

(i) if θ is atomic then θ^s is θ;

(ii) if θ is $\neg\psi(\exists x\psi, \forall x\psi)$ then θ^s is $\neg\psi^s(\exists x\psi^s, \forall x\psi^s)$;

(iii) if θ is $\bigwedge_{\xi<\kappa} \theta_\xi(\bigvee_{\xi<\kappa} \theta_\xi)$ then θ^s is $\bigwedge_{\xi\in s} \theta^s(\bigvee_{\xi\in s} \theta^s_\xi)$.

Notice that θ^s is always a formula of $L_{\omega_1\omega}$, and that if θ is a formula of $L_{\omega_1\omega}$, then θ^s is θ for almost all s.

By induction on formulas, using the Lemma giving properties of D, we obtain the following generalized Löwenheim-Skolem theorem.

THEOREM 1. *Assume that $|A| \leqq \kappa$, and let θ be a sentence of $L_{\kappa^+\omega}$. Then $\mathfrak{A} \models \theta$ iff $\mathfrak{A}_s \models \theta^s$ for almost all s.*

As immediate consequences we obtain result (B) above and the following:

COROLLARY. *Assume θ can be written in negation-normal form (that is, only atomic subformulas are negated) without uncountable disjunctions. Then $\models\theta$ iff $\models\theta^s$ for almost all s. In particular, if σ and $\psi_\xi(x)$ belong to $L_{\omega_1\omega}$ $(\xi < \kappa)$, then $\models\sigma \to \exists x \bigwedge_{\xi<\kappa} \psi_\xi(x)$ iff $\models\sigma \to \exists x \bigwedge_{\xi\in s} \psi_\xi(x)$ for all countable $s \subseteq \kappa$.*

Another consequence of Theorem 1 is the following characterization of $\equiv_{\infty\omega}$ which generalizes Scott's Isomorphism Theorem (see [**1**]).

THEOREM 2. *Assume that $|A|, |B| \leqq \kappa$. Then*

(i) $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A}_s \cong \mathfrak{B}_s$ for almost all s;

(ii) $\mathfrak{A} \not\equiv_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A}_s \not\cong \mathfrak{B}_s$ for almost all s.

To prove Theorem 2 it is enough to prove both of the implications from left to right. For (i) this is not difficult, using the standard back-and-forth properties of $\equiv_{\infty\omega}$ (see [1]). For (ii) this is immediate from Theorem 1. S. Shelah has observed that (ii) also follows from a game-theoretic characterization of $\equiv_{\infty\omega}$ and the Gale-Stewart theorem that open games are determined.

As might be expected from Theorem 2, reduced products of countable models modulo the filter D can also be used to characterize $\equiv_{\infty\omega}$.

DEFINITION. (a) L^* is the expansion of L formed by adding a new predicate $P_\neg$ for every predicate P (including $=$) of L. (b) If $\mathfrak{A}$ is an L-model then $\mathfrak{A}^*$ is its expansion to L^* satisfying

$$\forall v_0 \cdots v_k\, [P\neg(v_0, \cdots, v_k) \leftrightarrow \neg P(v_0, \cdots, v_k)].$$

(c) If $\mathfrak{A}'$ is an L^*-model and $\mathfrak{B}' \subseteq \mathfrak{A}'$, then $\mathfrak{B}'$ is *strongly maximal* in $\mathfrak{A}'$ if B' is a maximal subset of A' satisfying $\forall xy\, (x = \neg y \leftrightarrow \neg x = y)$.

THEOREM 3. *Assume that* $|A|, |B| \leqq \kappa$. *The following are equivalent*:
(i) $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$;
(ii) $\Pi\mathfrak{A}_s^*/D \cong \Pi\mathfrak{B}_s^*/D$;
(iii) $\mathfrak{B}^*$ *is isomorphic to a strongly maximal submodel of* $\Pi\mathfrak{A}_s^*/D$.

The implication from (i) to (ii) is immediate from Theorem 2(i). The implication from (ii) to (iii) is not difficult, using the Lemma giving properties of D. To show that (iii) implies (i) we first show that $\{s: \mathfrak{A}_s \not\cong \mathfrak{B}_s\} \notin D$, and then use Theorem 2(ii) to conclude that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. Examples show the * is necessary for (ii) to imply (i).

We also obtain results analogous to Theorems 2 and 3 for embeddability in place of isomorphism.

2. $L^P(\omega)$ **and closed classes.**

DEFINITION. Let $\boldsymbol{K}$ be a class of models closed under isomorphism.

(a) $\boldsymbol{K}$ is *closed* if: $\mathfrak{A} \in \boldsymbol{K}$ iff $\mathfrak{A}_0 \in \boldsymbol{K}$ for almost all countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$.

(b) $\boldsymbol{K}$ is *closed downward* if: whenever $\mathfrak{A} \in \boldsymbol{K}$, then $\mathfrak{A}_0 \in \boldsymbol{K}$ for almost all countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$.

Classes which are closed downward satisfy a downward Löwenheim-Skolem theorem, while closed classes also satisfy an upward theorem. Theorem 1 implies that Mod(σ) is closed if σ is a sentence of $L_{\omega_1\omega}$, and Theorem 2 implies that closed classes are closed under $\equiv_{\infty\omega}$. A closed class is uniquely determined by the countable models in it, and hence, there are 2^{2^ω} different closed classes. If $\boldsymbol{K}$ and its complement are closed downward then $\boldsymbol{K}$ is closed, but the converse fails. If $\boldsymbol{K}'$ is a class of L'-models which is closed downward, then $\boldsymbol{K}' \upharpoonright L$ (= the class of all reducts of models in $\boldsymbol{K}'$ to L) is also closed downward. Therefore Mod(σ') $\upharpoonright L$ is closed downward, but not generally closed, for any sentence σ' of $L'_{\omega_1\omega}$.

We define $L^p(\omega)$ to be the class of formulas of Keisler's $L(\omega)$ (from [2]) which can be put in prenex form. Thus, $\sigma \in L^p(\omega)$ iff σ is equivalent to some $(Q_n v_n)_{n<\bar{\omega}}\chi$, where χ is a quantifier-free formula (in countably many variables). Therefore $\sigma \in L^p(\omega)$ if σ is a formula of $L_{\omega_1\omega}$ or a universal or existential sentence of $L_{\omega_1\omega_1}$. If σ is a sentence of $L^p(\omega)$ then $\text{Mod}(\sigma)$ is closed downward, but not generally closed. Most of the interest of $L^p(\omega)$ stems from:

THEOREM 4. *If $\boldsymbol{K}$ is closed then $\boldsymbol{K} = \text{Mod}(\sigma)$ for some sentence σ of $L^p(\omega)$.*

COROLLARY 1. *If $\boldsymbol{K}$ is closed downward then there is a sentence σ of $L^p(\omega)$ such that $\boldsymbol{K} \subseteq \text{Mod}(\sigma)$, and $\boldsymbol{K}$ and $\text{Mod}(\sigma)$ contain precisely the same countable models.*

The intersection of two classes which are closed downward is either empty or contains a countable model. Hence, Corollary 1 implies a separation result for disjoint classes closed downward, a particular case of which is the following interpolation theorem for $L^p(\omega)$.

COROLLARY 2. *Let L_1 and L_2 be countable languages whose intersection is L. Let $\theta \in L_1^p(\omega)$ and $\psi \in L_2^p(\omega)$, and assume that $\models \theta \rightarrow \neg\psi$. Then there is some $\sigma \in L^p(\omega)$ such that $\models \theta \rightarrow \sigma$ and $\models \sigma \rightarrow \neg\psi$.*

The case of Corollary 1 where $\boldsymbol{K} = \text{Mod}(\bar{\Sigma}) \upharpoonright L$ for some set $\bar{\Sigma}$ of finitary sentences is due to Svenonius [3]. Corollary 2 is essentially due to Takeuti (see the next section). Even if θ and ψ are also in $L_{i_{\omega_1\omega_1}}$, Malitz's example (given in [4]) shows that the interpolant σ need not be in $L_{\infty\omega_1}$.

The logic $L^p(\omega)$ also admits some preservation theorems.

DEFINITION. (a) $\mathfrak{A}$ is the *β-union* of a nonempty set S of submodels of $\mathfrak{A}$ (where β is any cardinal > 0) if every subset of A of power less than β is contained in the universe of some model in S.

(b) $(\forall^n\exists)^p(\omega)$ is the set of all sentences of $L^p(\omega)$ of the form $\forall x_0 \cdots x_{n-1} \exists y_0 \cdots y_k \cdots \chi$, where χ is quantifier-free.

THEOREM 5. (i) *$\boldsymbol{K}$ is closed downward and closed under $(n+1)$-unions iff $\boldsymbol{K} = \text{Mod}(\theta)$ for some $\theta \in (\forall^n\exists)^p(\omega)$.*

(ii) *$\boldsymbol{K}$ is closed downward and closed under ω-unions iff $\boldsymbol{K} = \text{Mod}(\bigwedge_n \theta_n)$ where $\theta_n \in (\forall^n\exists)^p(\omega)$ for all n.*

A sentence whose negation is in $(\forall^0\exists)^p(\omega)$ is *universal*. Case $n = 0$ of Theorem 5 then implies: $\boldsymbol{K}$ is closed and closed under submodels iff $\boldsymbol{K} = \text{Mod}(\theta)$ for some universal θ of $L^p(\omega)$. This is a different formulation of a theorem of Tarski [5].

3. **Generalizations to uncountable models.** If λ is any infinite cardinal and $\kappa > \lambda$ we can define a filter over $\mathscr{P}_\lambda^+(\kappa)$ analogously to §1 and

obtain a notion of "almost all" subsets of κ of power at most λ. Most of the results of the preceding sections have analogues here, especially if $\lambda^{\underline{\lambda}} = \lambda$. Some of them are of less interest, however, due to the failure of the Isomorphism Theorem for models of uncountable regular power (see [**2**]). We do obtain the following interpolation theorem generalizing Corollary 2 of §2. $L^p(\lambda)$ is the set of all formulas equivalent to some $(Q_\xi v_\xi)_{\xi<\lambda}\chi$, where χ is quantifier-free. Hence every formula of $L_{\lambda^+\lambda}$ belongs to $L^p(\lambda)$.

THEOREM 6. *Let L_1 and L_2 be languages, with at most $\lambda^{\underline{\lambda}}$ nonlogical symbols, whose intersection is L. Assume that $\theta \in L_1^p(\lambda)$, $\psi \in L_2^p(\lambda)$, and $\models \theta \to \neg\psi$. Then there is some $\sigma \in L^p(\lambda^\lambda)$ such that $\models \theta \to \sigma$ and $\models \sigma \to \neg\psi$.*

The case where θ and ψ belong to $L_{i_\lambda^+\lambda}$ was proved (syntactically) by Takeuti [**4**], in response to Malitz's examples of implications which have no interpolants in any $L_{\kappa\lambda}$. This case in fact implies the above form of the theorem, but Takeuti does not obtain the general results from which we derive it.

REFERENCES

1. C. C. Chang, "Some remarks on the model theory of infinitary languages," in *The syntax and semantics of infinitary languages*, Springer-Verlag, Berlin and New York, 1968, pp. 36–63.

2. H. J. Keisler, "Formulas with linearly ordered quantifiers," in *The syntax and semantics of infinitary languages*, Springer-Verlag, Berlin and New York, 1968, pp. 96–130.

3. L. Svenonius, *On the denumerable models of theories with extra predicates*, Proc. Internat. Sympos. Theory of Models (Berkeley, 1963), North-Holland, Amsterdam, 1965, pp. 376–389. MR **35** #42.

4. G. Takeuti, "A determinate logic," in *The syntax and semantics of infinitary languages*, Springer-Verlag, Berlin and New York, 1968, pp. 237–264.

5. A. Tarski, *Remarks on predicate logic with infinitely long expression*, Colloq. Math. **6** (1958), 171–176. MR **20** #6351.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104