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EXPONENTIATION OF CERTAIN QUADRATIC INEQUALITIES FOR SCHLICHT FUNCTIONS[1]

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The Grunsky inequalities characterize the analytic functions that are univalent. Theorem 1 gives a new set of inequalities which appear to be the result of exponentiating the Grunsky inequalities for functions on the unit disc.

THEOREM 1. *If $f(z) = z + a_2z^2 + \cdots$ is a one-to-one, analytic function on $\{z:|z| < 1\}$, then*

$$(1) \qquad \left|\sum_{\nu,\mu=1}^{n} \alpha_\nu\alpha_\mu \frac{f(z_\nu)}{z_\nu}\frac{f(z_\mu)}{z_\mu}\frac{z_\nu - z_\mu}{f(z_\nu) - f(z_\mu)}\right| \leqq \sum_{\nu,\mu=1}^{n} \alpha_\nu\bar{\alpha}_\mu \frac{1}{1 - z_\nu\bar{z}_\mu}$$

for all z_ν in the unit disc and all complex numbers α_ν for $n = 1, 2, \ldots$. For $z_\nu = z_\mu$ replace $(z_\nu - z_\mu)/(f(z_\nu) - f(z_\mu))$ by $1/f'(z_\nu)$.

This theorem can be proved by an extension by Goluzin's method [**2**] of using Löwner's differential equation [**4**] to prove the Grunsky inequalities. Using (1), it is easy to find the bounds on the coefficients of the inverse function $f^{-1}(w)$ for all functions f as described in Theorem 1. (This problem was first solved by Löwner [**4**].)

By the same method, the following theorem can be proved.

THEOREM 2. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is a one-to-one, analytic function on $\{z:|z| < 1\}$, then*

$$(2) \qquad \sum_{\nu,\mu=1}^{n} \alpha_\nu\bar{\alpha}_\mu \left|\frac{f(z_\nu) - f(z_\mu)}{z_\nu - z_\mu}\frac{1}{1 - z_\nu\bar{z}_\mu}\right| \geqq \left|\sum_{\nu=1}^{n} \alpha_\nu \left|\frac{f(z_\nu)}{z_\nu}\right|\right|^2$$

and

$$(3) \qquad \sum_{\nu,\mu=1}^{n} \alpha_\nu\bar{\alpha}_\mu \left|\frac{f(z_\nu) - f(z_\mu)}{z_\nu - z_\mu}\frac{1}{1 - z_\nu\bar{z}_\mu}\right|^2 \geqq \left|\sum_{\nu=1}^{n} \alpha_\nu \left|\frac{f(z_\nu)}{z_\nu}\right|^2\right|^2$$

for all z_ν in the unit disc, for all complex numbers α_ν and $n = 1, 2, \ldots$. For $z_\nu = z_\mu$ replace $(f(z_\nu) - f(z_\mu))/(z_\nu - z_\mu)$ by $f'(z_\nu)$.

From (2) it follows that if the coefficients of f are all real, then $a_1 + a_3 + \cdots + a_{2n-1} \geqq a_n^2$ and consequently $|a_n| \leqq n$ for $n = 1, 2, \ldots$. (That the

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Bieberbach conjecture holds for functions with real coefficients was first proved by Dieudonné [**1**] and Rogosinski [**6**].)

From (3) it follows that

$$\sum_{k=1}^{n} k|a_k|^2 + \sum_{k=n+1}^{2n-1} (2n-k)|a_k|^2 \geqq |a_n|^4 \tag{4}$$

and consequently $|a_n| \leqq (7/6)^{1/2}n$ for $n = 1, 2, \ldots$. The constant $(7/6)^{1/2}$ is not the smallest that follows from inequality (3), but this estimate already compares favorably with the best previous result $|a_n| \leqq (1.243)n$ obtained by Milin [**5**].

From (3) also follows a more general inequality than (4) which implies that $\limsup_{n\to\infty}|a_n|/n < 1$, except in case $f(z) = z/(1 - e^{i\theta}z)^2$. (This theorem was first proved by Hayman [**3**].)

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