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BOOK REVIEWS

Abstract Harmonic Analysis; I, II, by E. Hewitt and K. A. Ross. Springer-Verlag, New York, Heidelberg, Berlin (1963; 1969).

Abstract harmonic analysis is concerned with the theory of Fourier series and integrals in the context of topological groups.

As would be expected even from such a summary description "abstract" theories and "classical" theories are intimately connected. Abstract harmonic analysis cannot replace classical Fourier analysis but it is now almost impossible to work in the latter subject without having in mind the "abstract" developments.

The relationship between classical theories and new theories however is by no means clear. For one thing there is no general agreement as to the degree of generality in which it is most fruitful to work for each problem. Following the discovery of an invariant measure on all locally compact topological groups, and with the development of the theory of Banach algebras, it became apparent that a large portion of classical harmonic analysis could be done in the context of locally compact abelian (LCA) groups.

The very satisfactory duality theory for LCA groups and the existence of powerful structure theorems contributed to make this development possible. The theorems of Katznelson (and others) on functions that operate in $L^1(G)$, the proof, by Malliavin, of the impossibility of spectral synthesis, the characterization established by P. J. Cohen of the idempotent measures were all conceived in the context of LCA groups. Even the classical theory of functions on the disk was fruitfully extended to the case of LCA groups, with appropriate additional hypotheses.

These developments are described in W. Rudin's monograph, *Fourier analysis on groups*, which appeared in the early sixties and provided further stimulus for research in this area.

The situation is different for harmonic analysis on noncommutative groups. Duality theory for nonabelian groups is based on the concept of irreducible representations, but on the other hand, the task of describing the irreducible representations of noncommutative groups, even finite groups, seems formidable.

Nevertheless, a considerable amount of research has been done in Fourier analysis on noncommutative groups. Some researchers are concerned with particular groups, such as $SL(2)$ and $U(2)$, of which the representations are carefully analyzed and described in order to develop the theory; but others are attempting to develop general methods which will work for wide classes of topological groups, such as the class of all "amenable" groups or all compact groups.

The book under review can also be considered as an attempt to provide the basic tools to do research in harmonic analysis well beyond the boundaries of LCA groups; and the class of nonabelian groups which the authors choose to emphasize is that of compact groups. Of course the two volumes contain an incredible amount of information on general topological groups as well and they present, in an original fashion and in full detail, many results and techniques which are proper of harmonic analysis on LCA groups; and indeed the first volume contains a very detailed analysis of the structure of LCA groups with a wealth of illuminating examples.

This reviewer feels however that perhaps the most important contribution of the second volume of this treatise is that of opening a wide avenue for research in harmonic analysis on noncommutative compact groups.

Notwithstanding the great number of research papers which are published on this subject, compact noncommutative groups have not been in the limelight of harmonic analysis. Research on this topic has been rather fragmented and without a definite and clear direction. It is to be expected that the publication of this book will alter this situation considerably and will also induce many students of commutative harmonic analysis to work on noncommutative groups. It is probable in any case that many people will be surprised by the extent to which harmonic analysis on noncommutative compact groups is developed in the book under review. It is not surprising that the methodic and intelligent application of techniques of "abstract analysis" should provide generalizations of results from the theory of Fourier series. But the book under review offers to the reader a number of specific tools which are typical of harmonic analysis such as the constructions of special kernels and special trigonometric polynomials and the technique of randomizing the coefficient of a Fourier series. In fact what might come as a surprise to many is that some of these tools can be constructed for all compact groups, often by performing the necessary calculations only in the case of particular compact subgroups of the unitary group.

This treatise however is by no means a monograph on compact groups. Its encyclopedic character is especially evident in the first volume which, in the authors' words, "gives all of the structure of topological groups needed for harmonic analysis as it is known to us; it treats integration on locally compact groups in detail; it contains an introduction to the theory of group representations." In fact after a preliminary chapter this volume contains in Chapter 2 a treatment of the general theory of topological groups, including the structure of locally compact abelian groups. Especially interesting features of this chapter are a discussion of linear groups and the exponential map, and a section on "special locally compact abelian groups."

The theory of integrations on locally compact topological spaces is treated very carefully and completely in Chapter 3. A full and detailed presentation of complex measures is also given in this chapter.

Chapter 4 is dedicated to the Haar integral and other invariant functionals. The existence of the Haar integral is proved without recourse to the axiom of choice. Invariant means on the space of bounded functions on a group are then discussed and it is proved that such means fail to exist if the group contains a free subgroup on two generators. Finally invariant means on almost periodic functions are briefly treated.

In Chapter 5 the book takes up the subject matter proper of harmonic analysis with an introduction to convolution and group representations. A very general and useful definition of convolution is given, following the work of Hewitt and Zuckerman. Onde finds a detailed treatment of $M(G)$ as a convolution algebra, but also other examples of convolution algebras are considered, such as the dual space $C^*_{ru}(G)$, of the right uniformly continuous functions on a locally compact group. (The reader will also find here the curious fact that the convolution algebra $C^*_{ru}(G)$ might not be commutative even when G is commutative.) Convolutions of functions and measures are then studied in detail. Finally representations of groups and algebras are introduced leading up to the study of unitary representations of locally compact groups. The Gelfand-Raikov theorem and the fact that irreducible representations of compact groups are finite dimensional are proved in this chapter.

Chapter 6 deals with the character group and the duality theorem of LCA groups. The authors choose to prove the Pontryagin-van Kampen duality theorem using the structure theory developed in Chapter 2. Of course once this theorem is proved, the structure of LCA groups can be analyzed more closely. This is done in the rest of this chapter, in some sixty pages, which contain an incredible amount of useful information. The authors give general theorems on the structure of certain classes of groups, and also analyze in great detail particular important examples. In the reviewer's opinion the material presented here is of the greatest importance. In fact it is becoming more and more clear that the "classical groups" of the torus, the real line and the integers do not always provide the context in which all theorems are proved most naturally. For instance discrete torsion groups might be more tractable than the integers to prove certain theorems which can be later "generalized" to the integers. A knowledge of special groups is therefore an essential part of what "a modern harmonic analyst should know."

The second volume of this treatise begins with Chapter 7. This is a chapter more than 200 pages long, by itself a monograph on the theory of representations of compact groups. It is divided into three sections: §27 contains the general theory of unitary representations. It includes the Peter-Weyl theorem and the theorem on the extension of representations of subgroups. In §28 the "dual object" of a compact group G, that is the space Σ of equivalence classes of irreducible representations of G, with its operations of tensor product and conjugation, is studied in detail and the duality between normal subgroups and their annihilators as subsets of Σ is carefully investigated. Finally the notion of Fourier transform for measures defined on a compact group is introduced. The treatment given here of this notion, based in part on the work of R. Kunze, is very complete and satisfactory. The authors define also the spaces $\mathfrak{E}_p$ which are the analogues of the l^p spaces and the Fourier transform is quite naturally a map from the space of measures into l^∞. Thus the stage is set for the Hausdorff-Young theorem which is proved in the following chapter. §29 contains various special topics, including the computation of important integrals involving unitary groups. The results of these computations are used in the theory of multipliers which is treated in a later chapter. A complete analysis of the irreducible unitary representations of $SU(2)$ is also presented in this section In §30 one finds a very original presentation of the duality theorems of Tannaka and Krein. This is an extremely interesting section: even though, as the authors point out, only a fraction of the material presented in this section is used in the rest of the book, the topics treated here are most suitable to give the reader an idea of "what the real difference is" between commutative and noncommutative harmonic analysis.

Chapter 8 is entitled "Fourier transforms." It consists of three sections. Transforms of L^p functions for LCA and compact groups are studied in §31. The treatment includes the Hausdorff-Young theorem and the study of special approximating kernels. In §32 positive definite functions are defined and studied in general LC groups. Cohen's factorization theorem with all its later versions and extensions is presented here and is used to prove the theorem of Varopoulos on positive functionals on a Banach algebra with involution.

§32 contains Bochner's theorem on positive definite functions and several of its applications: the construction of all $*$-representations of L of an LCA group; Stone's theorem on unitary representations of LCA groups.

Compact groups (generally noncommutative) reappear in Chapter 9. First of all the authors set the stage by defining and studying $\mathfrak{R}(G)$, the algebra of continuous functions with absolutely convergent Fourier series. The definition is based on the work of Krein. It turns out, quite naturally, that $\mathfrak{R}(G) = L^2(G) * L^2(G)$ and that $\mathfrak{E}_1$ is isometric to $\mathfrak{R}(G)$. As the authors remark, there is hardly a question which has been posed for the algebra of continuous periodic functions, with absolutely convergent Fourier series which is not meaningful and interesting for $\mathfrak{R}(G)$. "Multipliers over compact groups" are treated in §35 and §36. It is known that many problems in classical harmonic analysis can be formulated as multiplier problems, or dually, as convolution problems. This is also the case for compact noncommutative groups. Indeed a careful application of functional analytic techniques leads to extensions of many classical results to this more general context. These extensions are presented in §35. With §36, the authors, using results already established in §29, begin a careful study of unitary multipliers, based on the analysis of the infinite product of unitary groups. In fact without explicitly mentioning it, the authors develop for noncommutative groups, parts of the theory of random Fourier series, beginning with a careful study of the analogues of the Steinhaus series (sums of independent random variables equidistributed in $(0, 2\pi)$). It turns out that the inequalities concerning L^p norms of Steinhaus' series hold for their analogues, but their proofs lie much deeper, so that in the end the results suggest as many interesting problems as they solve. Many results concerning multipliers are summarized in a table at the end of §36; in this table the reader will find many theorems familiar from classical Fourier analysis but also many interesting surprises. The subject of lacunary Fourier series is connected very strongly with that of multipliers and random Fourier series. In this book it is treated in §37. The notions of Sidon set and $\Lambda(p)$ set are defined for subsets of the "dual object" of G and their properties analyzed and compared. A number of interesting constructions relating to lacunary sets, mostly for noncommutative groups are also discussed in this section. A section on the ideal theory for the algebras $L^p(G)$ (under convolution) completes Chapter 9. This section besides being of interest in its own right provides an introduction to the problems treated in the following chapter.

Chapter 10 contains a discussion of the ideal theory of regular semisimple commutative Banach algebras (§39) and of the problem of spectral synthesis for LCA groups. The authors present a proof of Malliavin's theorem based on the work of Varopoulos. They choose however not to start from Schwarz's result on the failure of spectral synthesis for R^3 but, following Katznelson, they begin by proving Malliavin's theorem for the Cantor group.

Chapter 11 contains two special topics. The first is the subject of §43 and it is an elegant and interesting study of the functions for which the Hausdorff-Young inequality is in fact an equality. The last section (§44) contains results on pointwise summability of Fourier transforms. The authors basically show, following the work of Edwards and Hewitt, that there exist central approximating kernels $K_{m,n}$ such that, for $f \in L^1(G)$ (G a LCA or a compact group), $\lim_m \lim_n K_{m,n} * f(x) = f(x)$, almost everywhere. The presence of a double limiting process makes the theorem generally different from the classical one but nevertheless it has already found important applications (one is given in (36.18)). In a number of special cases, which include all compact Lie groups, it is possible to find singly indexed approximating kernels, thus reproducing completely the classical situation. Quite apart from the theorem just mentioned, this reviewer believes that the results proved and the techniques developed in §44 will be of great importance in the development of harmonic analysis on general groups. This section in fact provides analogues of many classical constructions, such as the Dirichlet and Féjèr kernels and discusses extensions of theorems such as the Hardy-Littlewood maximal theorem. One of the most interesting outstanding problems of the theory of Fourier series on compact groups is that of extending the Paley-Littlewood theorem and its consequences, at least to some class of noncommutative groups. It is not unlikely that the techniques developed here might contribute in a significant way to the solution of this problem.

In conclusion the treatise under review seems to be destined to stimulate significant new research on harmonic analysis on general groups. As to the second volume this is especially true for what concerns noncommutative groups. This reviewer believes in particular that, quite apart from the new outlook and organization of the subject, and the many new results contained in the book, important tools for further research on compact groups are offered especially in §§29, 36 and 44, which should be carefully studied by anybody wanting to pursue research on Fourier series on noncommutative groups.

A few comments should be added as to the general readability of this treatise. The authors made a serious attempt to include details of proofs and to make the material available also to the inexperienced reader. There is no doubt however that the nonspecialist will find this book difficult to read. The excellent indices and the careful cross references may not be enough to help the inexperienced reader to wade through the enormous mass of information offered in the book, while it is perhaps unreasonable to expect such a reader to go through the book from cover to cover. Perhaps the titles of the subsections could have been included in the table of the contents; even though titles such as "Definitions and discussion" would not really be of much help. But probably the greatest obstacle in reading this treatise, especially the second volume, is that, without knowing the problems of classical Fourier series, it is just too difficult to appreciate and understand fully seemingly tedious computations. For example it is difficult to imagine anyone reading with interest the proof of Lemma (36.17) without having studied Zygmund's beautiful result (Lemma (8.3), Chapter V, of his treatise) of which Lemma (36.17) is an extension. It is certainly convenient for the specialist to have the theorems stated in their full generality and to be able to find computations as complete as those given in §29, but for the less experienced it would have been easier if certain results had been obtained as directly as possible without exploring "all the interesting byways."

Nevertheless, the reading of this treatise, no matter how difficult, will always be a rewarding experience for anybody interested in abstract harmonic analysis.

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Introduction to the Theory of Partially Ordered Spaces by B. Z. Vulikh, translated by Leo F. Boron. Gordon and Breach, New York, 1967. xv + 387 pp.

This title first appeared in Russian in 1961; the present translation appeared in print in 1967, and the reviewer's copy appeared in his mail in 1971. There is little difference perceptible between the reviewer's copies of the English and the Russian editions, so one might at first suppose that this book is more an historical document than a work of present interest. It is pleasant to report that this supposition is false; even though the book includes none of the beautiful and far-reaching results that functional analysis in partially ordered spaces has produced since about 1957, its treatment has many virtues and its contents perennial interest. In this late review, then, it seems appropriate to try to place the work in perspective, to examine briefly what it does and at a little more length what it does not do.

Hewitt's review of the Russian edition (MR **24** #A3494) describes the book as "an updated, abridged, and elementary version of the monograph *Functional analysis in partially ordered spaces* (Russian) by Kantorovič, Vuli(k)h and Pinsker (GITTL, Moscow, 1950; MR **12**, 340)," and gives a chapter-by-chapter list of the contents. One may thus omit the list here, and expand on the description by saying that this is primarily a book about vector lattices, usually complete or σ-complete, which is not really a simple introduction to but a fairly complete account of their theory. The theory, which began its development in the middle thirties, was itself fairly complete by the middle fifties. Two rather old-fashioned characteristics of the theory are particularly notable: its preference for convergence defined in terms of the lattice-order relations to norms and topologies (only two of the thirteen chapters of the present book deal with normed, or even Fréchet, spaces), and its lack of negative surprises (representation theorems show that the examples of vector lattices that come naturally to one's mind are pretty typical, and when one needs a counterexample a fairly familiar