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A STRUCTURE THEOREM FOR COMPLETE NONCOMPACT HYPERSURFACES OF NONNEGATIVE CURVATURE

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The convexity theorem of Sacksteder-van Heijenoort [4] states that if M is a C^∞ n-dimensional $(n>1)$ complete orientable Riemannian manifold of nonnegative sectional curvature with positive curvature at one point, then every isometric immersion $x: M \to \boldsymbol{R}^{n+1}$ is an imbedding and $x(M)$ bounds an open convex subset of $\boldsymbol{R}^{n+1}$; furthermore M is diffeomorphic to either $\boldsymbol{R}^n$ or S^n (unit n-sphere). The purpose of this note is to announce a structure theorem that complements the above result of Sacksteder and van Heijenoort. Full details will appear in a forthcoming monograph on convexity and rigidity of hypersurfaces.

THEOREM. *Let M be a C^∞ hypersurface in $\boldsymbol{R}^{n+1}$ $(n>1)$ which is complete, noncompact, orientable with nonnegative sectional curvature, which is in addition all positive at one point, then*:

(1) *The spherical image of M in the unit sphere S^n has a geodesically convex closure, which lies in a closed hemisphere.*

(2) *The total curvature of M (cf. Chern-Lashof [2]) does not exceed one.*

(3) *M is a pseudograph (see below for definition) over one of its tangent planes.*

(4) *M has infinite volume.*

COROLLARY. *Suppose the sectional curvature of M is in fact everywhere positive, then*:

(5) *The spherical map is a diffeomorphism onto a geodesically convex open subset of S^n. Consequently the spherical image lies in an open hemisphere.*

(6) *Coordinates in $\boldsymbol{R}^{n+1}$ can be so chosen that M is tangent to the hyperplane $x_{n+1}=0$ at the origin, and there is a nonnegative strictly convex function (i.e. its Hessian is everywhere positive definite) $f(x_1, \cdots, x_n)$ defined in a convex domain of $\{x_{n+1}=0\}$ such that M is exactly the graph of f.*

REMARKS. (A) A C^∞ convex hypersurface M (i.e. M is the full boundary of an open convex set) in $\boldsymbol{R}^{n+1}$ is said to form a *pseudograph*

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over the tangent plane H if and only if:

(a) M lies above H, i.e. designating a closed half-space of H as being above H, we have that M lies in this half-space.

(b) Let $\pi: R^{n+1} \to H$ be the orthogonal projection and let $A = \pi(M)$. Then over the interior A° (of A as a subset of H), M is the graph of a C^∞ function.

(c) For every $a \in A - A^\circ$, $M \cap \pi^{-1}(a)$ is a closed semi-infinite straight line segment.

(d) Every hyperplane strictly above H intersects M at a diffeomorph of the unit $(n-1)$-sphere S^{n-1}.

(B) When $n=2$ and the curvature of M is everywhere positive, this theorem (as well as the theorem of Sacksteder-van Heijenoort) was first proved by Stoker [**5**].

(C) Assertions (2)–(6) above all follow from assertion (1). We actually prove a more general result than (1):

PROPOSITION. *Let C be an open convex subset of R^{n+1} $(n \geqq 1)$ with connected boundary M and let $\gamma: M \to S^n$ be the spherical map (in the sense of Alexandrov, see Busemann* [**1**]). *Then the closure of $\gamma(M)$ is geodesically convex.*

The proof of this Proposition is achieved quite simply by employing the concept of the *barrier cone* of a convex set. See Rockafellar [**3**].

(D) Neither (1) nor the Proposition is true if the word "closure" is deleted. (Cf. Busemann [**1**, p. 25, (4.4)].)

(E) The Proposition has applications in the theory of convex surfaces, e.g. Alexandrov's theory of spherical measures on an open convex surface (Busemann [**1**, p. 31]) or the rigidity and nonrigidity theorems of Pogorelov and Olovyanishnikov on open convex surfaces (Busemann [**1**, pp. 167–168]).

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