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ON THE SELBERG CONDITION FOR SUBGROUPS OF SOLVABLE LIE GROUPS

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1. **Introduction.** In [1], S. P. Wang uses the techniques of [2] to prove a converse to a Selberg lemma for solvable groups. In [3], the author gave an elementary proof of the main result of [2] using the semisimple splitting. It is quite natural to expect that the results of [1] should also have an elementary proof in terms of the semisimple splitting. We do so in this paper.

2. **Preliminaries.** Let S be a simply connected solvable analytic group with nil-radical H. Suppose we imbed S as a subgroup of $GL(n, R)$, the group of all n by n nonsingular real matrices. Let $\mathfrak{A}(S)$ denote the algebraic hull of S. Then we can write $\mathfrak{A}(S)=N\cdot T$, where N is the group of all unipotent matrices in $\mathfrak{A}(S)$ and T is a maximal abelian subgroup of semisimple matrices in $\mathfrak{A}(S)$. The relevant facts about algebraic groups can be found in [6]. In [3] we stated the following result primarily due to L. Auslander [4].

There exists an imbedding of S as a subgroup of $GL(n, R)$ satisfying the following properties.

1. $H\subset N$.
2. The projection mapping $P:\mathfrak{A}(S)\to N$ restricted to S defines a diffeomorphism of S onto N. We denoted the restriction of P to S by $n:S\to N$.

Denote the projection mapping of $\mathfrak{A}(S)$ into T by t. Let C be a closed subgroup of S. As we have seen in [3] we can choose T so that $t(C)\subset\mathfrak{A}(C)$. Let $C'=\mathfrak{A}(C\cap H)C$. Then C'/C is compact and C' is closed in S. From our choice of T and since $[\mathfrak{A}(C), \mathfrak{A}(C)]\subset\mathfrak{A}(C\cap H)$ it follows that $n(C')$ is a closed subgroup of N. It is easy to see that the following statements are equivalent.

1. S/C is compact.
2. S/C' is compact.
3. $N/n(C')$ is compact.
4. $N=\mathfrak{A}(n(C'))$.

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3. **Wang's Theorem A.** We shall assume the following two simple lemmas from [1].

LEMMA 3.1. *Let V be a finite-dimensional vector space over the reals, W a proper subspace of V and G a connected solvable subgroup of* GL(V). *Then there is a neighborhood Ω of the identity in G such that $\bigcup_{g\in\Omega} g(W) \neq V$.*

LEMMA 3.2. *Let X be a nonempty conic open subset of V and Ω a compact neighborhood of* 0 *in V. Then for every x in X, there is a positive number r such that $sx+\Omega\subset X$ for all $s\geqq r$.*

Let C be a closed subgroup of the simply connected solvable analytic group S. We shall assume the notation and conventions of §2.

We say that C has the Selberg property in S if and only if for any s in S and for any neighborhood Ω of 1 in S there exists u and v in Ω and an integer $l>0$ such that us^lv is in C.

THEOREM A. *Suppose that C has the Selberg property in S. Then S/C is compact.*

PROOF. By 1.2 of [5], C' has the Selberg property. Since S/C is compact if and only if S/C' is compact we can replace C by C' in our discussion. Thus without loss of generality we will assume throughout that the subgroup C under consideration has the addition property that $\mathfrak{A}(C\cap H)\subset C$. It follows that $n(C)$ is a subgroup of N. We must prove that $N=\mathfrak{A}(n(C))$.

Assume that N is abelian. Suppose that $N\neq\mathfrak{A}(n(C))$. Let Ω_1 be the set of all x in N such the euclidean absolute value of x is less than one. Let Ω_2 be the set of all x in N whose euclidean absolute value is less than one half. Since S normalizes N, by Lemma 1 it is easy to see that there is a compact symmetric neighborhood Ω of 1 in S such that

(a) $\bigcup_{u\in\Omega} u\mathfrak{A}(n(C))u^{-1}\neq N$.

It follows from elementary topological group theory techniques (see p. 95 of [7]) that Ω can be chosen with the addition properties that

(b) For all u in Ω, $u\Omega_2u^{-1}\subset\Omega_1$.

(c) $\Omega^2\subset n^{-1}(\Omega_2)$ where n is the homeomorphism of S onto N introduced in §2.

By [5], HC/H has the Selberg property in the vector group S/H. Thus by [5] again we have that S/HC^- is compact. It follows from our previous discussion that $N=H+\mathfrak{A}(n(C))$.

Let $X = N - \bigcup_{u\in\Omega} u\mathcal{A}(n(C))u^{-1}$. Then X is a nonempty open conic. It follows that there is an h in H such that $h^l + \Omega_1 \subset X$, for all positive integers l. Since C has the Selberg property in S there are elements u and v in Ω and a positive integer l such that uh^lv is in C. Thus $uh^lu^{-1}n(uv)$ is in $n(C)$. Note that by $n(uv)$ we mean the application of the projection map n to the product uv. From this equation we get that $h^lu^{-1}n(uv)u$ is in $u^{-1}n(C)u \not\subset X$. Our choice of Ω gives $u^{-1}n(uv)u \in \Omega_1$. Thus $h^l + \Omega_1 \not\subset X$, a contradiction.

Suppose N is not abelian. Let $[x, y]$ denote the commutator $xyx^{-1}y^{-1}$ of x and y. Denote the last nontrivial term in the lower central series of N by M. Using induction on the number of terms of the lower central series of N we can assume that the theorem holds in the group S/M. By [5], MC/M has the Selberg property in S/M. Thus $N = M + \mathcal{A}(n(C))$.

Let z and z' be in N. We can write $z = xy$ and $z' = x'y'$ where x and x' are in M and y and y' are in $\mathcal{A}(n(C))$. Since M is central in N, $[z, z'] = [y, y']$ is in $\mathcal{A}(n(C))$. Thus $M \subset [N, N] \subset \mathcal{A}(n(C))$.

4. **Wang's Theorem B.** We shall be satisfied with proving Wang's Theorem B for the special case of S simply connected.

THEOREM B. *Let S be a simply connected solvable analytic group, C a discrete subgroup of S such that S/C is compact, and Z the centralizer of C in S. Then Z is abelian.*

PROOF. Since Z commutes with C it commutes with $\mathcal{A}(C)$. Thus Z acts by the identity map on N. This easily implies that Z must be contained in N. Thus Z is abelian.

REFERENCES

1. S. P. Wang, *On S-subgroups of solvable Lie groups*, Amer. J. Math. **92** (1970), 389–397.

2. G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math (2) **60** (1954), 1–27. MR **15**, 853.

3. R. Tolimieri, *Applications of the semisimple splitting*, Bull. Amer. Math. Soc. **77** (1971), 275–280.

4. L. Auslander and J. Brezin, *Almost algebraic Lie algebras*, J. Algebra **8** (1968), 295–313. MR **37** #334.

5. A. Borel, *Density properties for certain subgroups of semi-simple groups without compact components*, Ann. of Math. (2) **72** (1960), 179–188. MR **23** #A964.

6. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200–221. MR **19**, 1181.

7. L. S. Pontrjagin, *Topological groups*, GITTL, Moscow, 1954; English transl., Gordon and Breach, New York, 1966. MR **17**, 171; MR **34** #1439.

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