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A MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS WITH NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS[1]

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We present a maximum principle in integral form for optimal control problems whose system equations involve delays in the state and delays in the derivative of the state. The results are obtained for a very general class of neutral functional differential equations which includes as a special case the systems

$$\dot{x}(t) + A\dot{x}(t-h) = Bx(t) + Cx(t-h) + Du(t),$$

which have been studied extensively (as in [4]) and arise in many applications. The class of control problems considered include problems for which one wishes to minimize $\int_0^T x^2(t)\,dt$ while requiring that $u(t) \in U \subset R^n$, $t \in [0, T]$, and either $x|_{[T-h,T]}$ lie in a manifold in $AC([T-h, T], R^n)$ or $x(t) = \zeta(t)$ on $[T-h, T]$, ζ a fixed absolutely continuous function. These functional boundary conditions arise naturally since the "state" in neutral systems of the above type is a point in $AC([-h, 0], R^n)$.

Let α_0, t_0, and a be fixed in R with $-\infty < \alpha_0 < t_0 < a < \infty$, $I = [\alpha_0, a)$, $I' = [t_0, a)$. For x continuous on I and t in I', the notation $F(x(\cdot), t)$ will mean F is a functional in x, depending on any or all of the values $x(\tau)$, $\alpha_0 \leqq \tau \leqq t$. For $t \in I'$, let

$$D(x(\cdot), t) = x(t) - \sum_{l=1}^{p} a_l(t)x(h_l(t)) - \int_{\alpha_0}^{t} d_\theta[\nu(t, \theta)]x(\theta).$$

Assume $a_l: I' \to R^{n^2}$ is continuous and of bounded variation, $l = 1, \cdots, p$; $h_l: I' \to R$ is continuous and strictly increasing, and there exists $\Delta > 0$ such that $\alpha_0 \leqq h_l(t) < t - \Delta$, $l = 1, \cdots, p$. Let $\nu(s, \cdot): [\alpha_0, \infty)$

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$\to R^{n^2}$ be continuous and of bounded variation for all $s \in I'$, $\nu(s, \theta) = 0$ for $s - \Delta \leqq \theta$, and $v(\cdot, \theta)$ be of bounded variation over $[t_0, a)$ for all $\theta \in [\alpha_0, a)$. Assume there exists $L > 0$ such that $\int_{\alpha_0}^{\infty} d_\theta |\nu(t, \theta) - \nu(s, \theta)| \leqq L|t - s|$ for $s, t \in I$, and there is a continuous nondecreasing function $\delta: R \to R$, with $\delta(\Delta) = 0$, such that for all $t \in I'$, all $\epsilon \in [0, t - \alpha_0]$,

$$\operatorname*{var}_{s \in [t-\epsilon, t]} \nu(t, s) + \sum_{\{l: h_l(t) \geqq t - \epsilon\}} |a_l(t)| \leqq \delta(\epsilon).$$

Let $\mathfrak{U} \subset R^r$, $U: I' \to$ subsets of $\mathfrak{U}$. Define $\Omega = \{u: u$ measurable on I', $u(t) \in U(t)$ for $t \in I'\}$. Let $\mathcal{G}$ be an open, convex set in R^n and assume $f: C(I, \mathcal{G}) \times \mathfrak{U} \times I' \to R^n$ is C^1 in x, Borel-measurable in (u, t). Given compact $X \subset \mathcal{G}$, $u \in \Omega$, suppose there exists $m(t) \in L^1(I')$ so that for all $t \in I'$, $x \in C(I, X)$,

$$|f(x(\cdot), u(t), t)| \leqq m(t) \quad \text{and} \quad |df[x(\cdot), u(t), t; \psi]| \leqq m(t)\|\psi\|$$

if $\psi \in C([\alpha_0, t], R^n)$, where df is the Fréchet differential of f with respect to x. Given $\varphi \in C([\alpha_0, t_0], \mathcal{G})$, $u \in \Omega$, we define a solution on $[\alpha_0, \tau]$ of

(1) $x(\theta) = \varphi(\theta)$ on $[\alpha_0, t_0]$,

(2) $d(D(x(\cdot), t))/dt = f(x(\cdot), u(t), t)$,

to be a continuous function x satisfying (1) such that $D(x(\cdot), t)$ is absolutely continuous on $[t_0, \tau]$ and (2) is satisfied a.e. on $[t_0, \tau]$.

To simplify expressions, assume $t_1 \in (t_0, a)$ is fixed. Choose $h \in [0, t_1 - t_0]$ and let $L_{-\mu}, \cdots, L_0, \cdots, L_m$ be given C^1 functions from $C([\alpha_0, t_0], \mathcal{G}) \times C([t_1 - h, t_1], \mathcal{G})$ into R.

Problem 1. Minimize $J[\varphi, u] = L_0(x_{t_0}, x_{t_1-h,t_1})$ subject to $\varphi \in C([\alpha_0, t_0], \mathcal{G})$, $u \in \Omega$, x the solution to (1), (2) on $[\alpha_0, t_1]$, and $L_i(x_{t_0}, x_{t_1-h,t_1}) = 0$, $i = 1, \cdots, m$, $L_i(x_{t_0}, x_{t_1-h,t_1}) \leqq 0$, $i = -\mu, \cdots, -1$, where x_{t_0} is the restriction of x to $[\alpha_0, t_0]$, and x_{t_1-h,t_1} is the restriction of x to $[t_1 - h, t_1]$.

If (φ^*, u^*) is a solution of the problem with response z, let $\eta^*(t, \theta)$ be the $n \times n$ matrix function guaranteed by the Riesz representation theorem such that $\eta^*(t, t) = 0$, $\eta^*(t, \cdot)$ is left-continuous except at t, and

$$df[z(\cdot), u^*(t), t; \psi] = \int_{\alpha_0}^{t} d_\theta[\eta^*(t, \theta)]\psi(\theta), \qquad t \in [t_0, t_1].$$

Then it can be shown (see [**8**]) that there is a well-defined Borel-measurable function $Y(s, t)$ on $[\alpha_0, \infty) \times [t_0, t_1]$ given by $Y(s, t) = 0$, $s > t$, and $Y(t, t) = E$, the $n \times n$ identity matrix

$$Y(s, t) = E - \sum_{l=1}^{p} \int_{[s,t+\Delta]\cap h_l^{-1}[s,\infty)} d_\alpha[Y(\alpha, t)]a_l(\alpha)$$
$$+ \int_s^{t^+} d_\alpha[Y(\alpha, t)]\nu(\alpha, s) - \int_s^t Y(\alpha, t)\eta^*(\alpha, s)\, d\alpha, \qquad s < t,$$

and $Y(\cdot, t): [\alpha_0, \infty) \to R^{n^2}$ is left-continuous and of bounded variation for each $t \in [t_0, t_1]$.

THEOREM 1. *Let* (φ^*, u^*) *be a solution to Problem* 1 *with response* z. *In addition, assume that* $\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{t_1-h,t_1}; \cdot, \cdot] = 0$ *and* $\alpha^i \leqq 0$ *for* $i \in \{-\mu, \cdots, 0\}$ *implies that* $\alpha^i = 0$, $i = -\mu, \cdots, m$. *Then there exists a row n-vector function* $\psi(s)$ *defined for all* $s \geqq t_0$, *and real numbers* α^i, $i = -\mu, \cdots, m$, *such that*

(i) $\alpha^i \leqq 0$ *for* $i \leqq 0$, $\alpha^i = 0$ *for all* $i \in \{-\mu, \cdots, -1\}$ *such that* $L_i(z_{t_0}, z_{t_1-h,t_1}) < 0$, $\sum_{i=-\mu}^{m} |\alpha^i| > 0$.

(ii)
$$\psi(s) = \sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{t_1-h,t_1}; 0, Y_{t_1-h,t_1}(s, \cdot)],$$
$$\psi(t_0) = -\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{t_1-h,t_1}; Y_{t_0}(t_0, \cdot), 0],$$

$\psi(s)$ *is nonzero on a subset of* $[t_0, t_1]$ *of positive measure.*

(iii)
$$\int_{t_0}^{t_1} \psi(s) f(z(\cdot), u(s), s)\, ds \leqq \int_{t_0}^{t_1} \psi(s) f(z(\cdot), u^*(s), s)\, ds$$

for all $u \in \Omega$.

The proof uses the abstract maximum principle of Neustadt [**10**], and the idea of a quasi-convex family as developed by Gamkrelidze [**6**], Neustadt [**9**], and Banks [**1**]. The development of the required theory of neutral equations is essentially contained in Hale and Cruz [**7**].

Let L_0 be as before, and let ζ be a specified nonzero function in $C([t_1-h, t_1], \mathcal{G})$.

Problem 2. Minimize $J[\phi, u] = L_0(x_{t_0}, x_{t_0,t_1})$ subject to $\phi \in C([\alpha_0, t_0], \mathcal{G})$, $u \in \Omega$, x the solution to (1), (2) on $[\alpha_0, t_1]$, and

$$\int_{t_1-h}^{t_1} \zeta(t) \cdot [\zeta(t) - x(t)]\, dt \leqq 0,$$
$$\sup_{t \in [t_1-h, t_1]} \{x^2(t) - \zeta^2(t)\} \leqq 0.$$

Again, let (ϕ^*, u^*) be a solution with response z. Note that the last two conditions imply that $z(t)=\zeta(t)$ on $[t_1-h, t_1]$. Define $Y(s, t)$ as before.

THEOREM 2. *Let (ϕ^*, u^*) be a solution to Problem 2, with response z. In addition, assume*

(a) *for all $\tau\in[\alpha_0, t_1]$, all $\epsilon>0$, there exists $\rho(\tau, \epsilon)>0$ so that $|t-\tau|<\rho(\tau, \epsilon)$ implies $|\nu(s, t)-\nu(s, \tau)|<\epsilon$ for all $s\in[t_0, t_1]$.*

(b) $h_l(t)=t-\theta_l$, $l=1, \cdots, p$.

$$\text{(c)}\qquad \begin{aligned} &f(x(\cdot), u(t), t)\\ &\quad = \bar{f}\left(x(t), x(g_1(t)), \cdots, x(g_q(t)), \int_{\alpha_0}^{t} d_\theta[\bar{\eta}(t, \theta)]G(x(\theta)), u(t), t\right)\end{aligned}$$

where g_l is continuous and strictly increasing on $[t_0, t_1]$, $\alpha_0\leqq g_l(t)<t$, $l=1, \cdots, q$; $G\in C^1(R^n, R^n)$; $\bar{\eta}:[t_0, t_1]\times[\alpha_0, t_1]\to R^{n^2}$ is measurable, $\bar{\eta}(t, \theta)=0$ for $\theta\geqq t$, $\bar{\eta}(t, \cdot):[\alpha_0, t]\to R^{n^2}$ is continuous and of bounded variation, $\operatorname{var}_{s\in[\alpha_0,t]}\bar{\eta}(t, s)\leqq k(t)$ for all $t\in[t_0, t_1]$, $k\in L^1([t_0, t_1], R)$.

Then there exist row n-vector functions $\psi(s)$ and $\psi_1(s, t)$ defined for all $s\geqq t_0$, $t\in[t_1-h, t_1]$, and real numbers $\alpha^0, \alpha^{-1}, \alpha^{-2}$ such that

$$\text{(i)}\qquad \alpha^i\leqq 0,\qquad i=0, -1, -2,\qquad \sum_{i=-2}^{0}|\alpha^i|>0,$$

$$\text{(ii)}\quad \begin{aligned} \psi(s) &= \alpha^0 dL_0[z_{t_0}, z_{t_0,t_1}; 0, Y_{t_0,t_1}(s, \cdot)] - \alpha^{-1}\int_{t_1-h}^{t_1}\zeta(t)Y(s, t)\,dt,\\ \psi_1(s, t) &= 2\zeta(t)Y(s, t),\\ \psi(s) &= 0, \qquad\qquad \textit{for } s>t_1,\\ \psi_1(s, t) &= 0, \qquad\qquad \textit{for } s>t,\end{aligned}$$

$\alpha^0 dL_0[z_{t_0}, z_{t_0,t_1}; Y(t_0, \cdot), 0]+\psi(t_0)$ is in

$$\overline{\text{co}}[\{-\alpha^{-2}\psi_1(t_0, t):t\in[t_1-h, t_1]\}\cup\{-\alpha^{-2}\psi_1(t_0, t^-):t\in[t_1-h, t_1]\}]$$

where, for $S\subset R^n$, $\overline{\text{co}}(S)$ denotes the closed convex hull of S.

$$\text{(iii)}\quad \begin{aligned} \int_{t_0}^{t_1}\psi(s)f(z(\cdot), u(s), s)\,ds &\leqq \int_{t_0}^{t_1}\psi(s)f(z(\cdot), u^*(s), s)\,ds\\ &\quad + |\alpha^{-2}|\sup_{t\in[t_1-h,t_1]}\left\{\int_{t_0}^{t_1}\psi_1(s, t)[f(z(\cdot), u(s), s)\right.\\ &\qquad\qquad \left. - f(z(\cdot), u^*(s), s)]\,ds\right\}\end{aligned}$$

for all $u\in\Omega$.

The proof uses the same methods as the proof of Theorem 1. Similar results can be obtained for problems which include additional restraints L_i as in Problem 1; these may be utilized to define an initial manifold in $C([\alpha_0, t_0], \mathcal{G})$.

Both Theorem 1 and Theorem 2 hold for variable final time problems, although the statements of the results are more complicated due to the explicit dependence on t_1. In this case there is an additional transversality condition, and, in Problem 2, $\zeta(t-t_1)$ defined for $t\in[t_1-h, t_1]$ must be piecewise C^1. In either theorem, if ϕ is fixed, then the condition at t_0 in (ii) need not hold.

These results can be obtained, using the same methods, in cases where the dependence of f on the control is more complicated, as long as $\{f(x(\cdot), u(\cdot), t): u\in\Omega\}$ is a quasi-convex family of functions. Examples of such types of dependence are found in [**2**] and [**3**]. The results can also be derived under weaker differentiability assumptions on the restraint functionals (see [**9**], [**10**]). In some cases in Problem 1 the integral maximum principle is equivalent to a pointwise maximum principle (see [**6**], [**9**]).

This formulation includes the type of dependence $F(x(\cdot), t) = F(x_t, t)$ where $x_t\in C([-h, 0], R^n)$, $x_t(\theta) = x(t+\theta)$. Then $\alpha_0 = t_0 - h$, and the choice of h for the "length" of the terminal manifold is quite natural. The results clearly include necessary conditions for problems involving retarded functional differential systems with terminal function-manifold or fixed terminal function.

An important application of these results concerns linear hyperbolic partial differential equations with boundary conditions containing the controls. Using the method of characteristics, as explained in [**5**], one obtains a neutral functional differential equation with control. Solutions to the problem in this form may then be transformed into solutions of the original problem in terms of weak solutions to the partial differential equations.

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