

SOME RELATIONS BETWEEN THE METRIC STRUCTURE  
AND THE ALGEBRAIC STRUCTURE OF THE  
FUNDAMENTAL GROUP IN MANIFOLDS  
OF NONPOSITIVE CURVATURE

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**1. Introduction.** Let  $\hat{M}$  be a complete simply connected riemannian manifold of dimension  $n$  and sectional curvature  $K \leq 0$ . Working with closed geodesically convex subsets  $\emptyset \neq M \subset \hat{M}$ , we use the fact (see [4] or [6]) that  $M$  is a topological submanifold of  $\hat{M}$  of some dimension  $k$ ,  $0 \leq k \leq n$ , with totally geodesic interior  ${}^0M$  and possibly empty boundary  $\partial M$ . Note that  $M$  is star-shaped from every point, thus contractible, and in particular simply connected.

Consider a properly discontinuous group  $\Gamma$  of homeomorphisms of  $M$  that acts by isometries on  ${}^0M$ . If the elements of  $\Gamma$  satisfy the semisimplicity condition described below (automatic if  $\Gamma \backslash \hat{M}$  is compact), and if  $\Sigma$  is a solvable subgroup of  $\Gamma$ , then Theorem 1 exhibits a flat totally geodesic  $\Sigma$ -stable subspace  $E \subset M$ , complete in  $\hat{M}$ , such that  $\Sigma$  has finite kernel on  $E$  and  $\Sigma \backslash E$  is compact. Thus  $\Sigma$  is an extension of a finite group by a crystallographic group of rank  $\dim E$ . In particular,

- (i)  $\Sigma$  is finitely generated,
- (ii) if  $\Sigma \backslash M$  is compact, then  $M$  is a complete flat totally geodesic subspace of  $\hat{M}$ , and
- (iii) if  $\Gamma \backslash M$  is a manifold, then the image of  $E$  in  $\Gamma \backslash M$  is a compact totally geodesic euclidean space form.

Theorem 1 extends and unifies several results concerning the case where  $M = \hat{M}$  and  $\Gamma \backslash M$  is a compact manifold. Those results are the classical theorem of Preissmann [7] which says that if  $K < 0$  then every nontrivial abelian subgroup of  $\Gamma$  is infinite cyclic, Byers' extension [2] of Preissmann's theorem to solvable subgroups of  $\Gamma$ , the case [10] where the elements of  $\Sigma$  are bounded isometries of  $M$ , the case [11] where  $\Sigma$  is central in  $\Gamma$ , and the case [11] where  $\Gamma$  is nilpotent. Theorem 1 was known [9] in the case where  $M$  is riemannian symmetric and  $\Gamma \backslash M$  is compact. The case where  $M = \hat{M}$  and  $\Gamma \backslash M$  is

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a compact manifold recently was independently obtained by S. T. Yau [12].

Theorem 2 concerns the case where  $\Gamma \backslash M$  is compact,  $\partial M = \emptyset$ , and  $\Gamma$  is a direct product  $\Gamma_1 \times \Gamma_2$ . If  $\Gamma_1$  is centerless, it provides a  $\Gamma$ -invariant isometric splitting  $M = M_1 \times M'$  where  $\Gamma_2$  acts trivially on  $M_1$  and  $\Gamma_1 \backslash M_1$  is compact; if  $\Gamma$  is centerless it provides an isometric splitting  $\Gamma \backslash M = (\Gamma_1 \backslash M_1) \times (\Gamma_2 \backslash M_2)$ . The case where  $\Gamma \backslash M$  is a real analytic manifold recently was independently studied by B. Lawson and S. T. Yau [5].

The main results of this paper were presented at a seminar in honor of the hundredth birthday of É. Cartan, at the University of Paris in June 1970. Parts of this paper were the subject of lectures by the authors at Rutgers University, Columbia University, M.I.T., and S.U.N.Y. at Stony Brook, between October 1969 and April 1970.

**2. The flat subspace.** Let  $\rho$  denote metric distance on  $\hat{M}$ . Given  $\gamma \in \Gamma$  we have the displacement function  $\delta_\gamma(p) = \rho(p, \gamma p) \geq 0$  on  $M$ . It is known [1] that  $\delta_\gamma$  is concave, i.e. that  $\delta_\gamma \circ c$  is a real concave function for every geodesic  $c$  of  $M$ . Therefore, given any real number  $a$ , the set  $C_\gamma^a = \{p \in M \mid \delta_\gamma(p) \leq a\}$  is convex. If  $\gamma' \in \Gamma$  commutes with  $\gamma$  then  $\gamma'(C_\gamma^a) = C_\gamma^a$ ; in particular  $C_\gamma^a$  is  $\gamma$ -invariant. If  $\delta_\gamma$  assumes its infimum  $a_\gamma \geq 0$  on  $M$ , then we say that  $\gamma$  is *semisimple* and denote  $C_\gamma = C_\gamma^{a_\gamma}$  the minimal set of  $\gamma$ . This term is based on the symmetric space case [9, Lemma 3.6]; see [6]. Through every point  $p \in C_\gamma$  there is a  $\gamma$ -invariant geodesic line contained in  $C_\gamma$ , which may be reduced to a fixed point of  $\gamma$ . From now on we assume that every element of  $\Gamma$  is a semisimple isometry of  $M$ . It is classical [3] that this condition is satisfied if  $\Gamma \backslash M$  is compact.

**LEMMA 1.** *If  $\emptyset \neq C \subset M$  is closed, convex and invariant under  $\gamma \in \Gamma$  then  $C \cap C_\gamma \neq \emptyset$ .*

**PROOF.** First suppose that  $\gamma$  has no fixed point. We adapt an argument from [10, p. 16]. Let  $L \subset C_\gamma$  be an invariant line of  $\gamma$ . There exist  $p \in L$  and  $q \in C$  such that  $\rho(p, q) = \rho(L, C)$  because  $\gamma$  induces a nontrivial translation on  $L$ . Consider the quadrangle  $Q$  with vertices  $p, q, q' = \gamma q$  and  $p' = \gamma p$ . The geodesic segments from  $p$  to  $q$ , and from  $p'$  to  $q'$ , realize the distance from  $L$  to  $C$ . Since  $C$  is convex,  $Q$  has all interior angles  $\geq \pi/2$ . Using  $K \leq 0$  it follows that  $Q$  is a flat totally geodesic rectangle, so  $\delta_\gamma(q) = \rho(q, q') = \rho(p, p') = \delta_\gamma(p)$ . Thus  $q \in C \cap C_\gamma$ .

Next suppose that  $\gamma$  has a fixed point  $p$ . We imitate the preceding argument with  $p = p'$ . If  $q \neq q'$  then  $Q$  is the triangle with vertices  $p, q$  and  $q'$ .  $Q$  is flat and totally geodesic in  $M$  and the sum of its interior

angles is  $> \pi$ . Thus  $q = q'$ , so  $\delta_\gamma(q) = 0 = \delta_\gamma(p)$ , and  $q \in C \cap C_\gamma$  as before. Q.E.D.

LEMMA 2. *Let  $\gamma \in \Gamma$  have infinite order (i.e. no fixed points). Then the minimal set  $C_\gamma$  splits isometrically in  $M$  as  $D \times E$ , where  $D \subset M$  is convex and  $d \times E$  is the  $\gamma$ -invariant line through  $d \in D$ .*

PROOF. The arguments of [10] remain valid for the manifold  $C_\gamma$ , because  $C_\gamma$  is convex and  $\delta_\gamma > 0$  is constant on  $C_\gamma$ . Q.E.D.

THEOREM 1. *Let  $\Sigma$  be a solvable subgroup of  $\Gamma$ . Then  $M$  has a flat totally geodesic  $\Sigma$ -stable subspace  $E$  such that*

- (i)  $E$  is complete in  $\hat{M}$ ,
- (ii)  $\Sigma$  acts with finite kernel  $\Phi$  on  $E$ , and
- (iii)  $\Sigma \setminus E$  is compact.

*In particular,  $\Sigma$  is finitely generated and is an extension of the finite group  $\Phi$  by a crystallographic group of rank  $= \dim E$ .*

PROOF. First suppose  $\Sigma$  abelian. If  $S \subset \Sigma$  denote  $C_S = \bigcap_{\sigma \in S} C_\sigma$ . Let  $T$  be the torsion subgroup of  $\Sigma$ . If  $T \neq \{1\}$  let  $1 \neq \tau \in T$ ; then  $\dim C_\tau < \dim M$  and  $T$  acts as a torsion abelian group on  $C_\tau$ . By induction on dimension now  $T$  has a fixed point on  $C_\tau$ . Thus  $T$  is finite and  $C_T \neq \emptyset$ . If  $\sigma_1 \in \Sigma - T$  then  $C_{\sigma_1}$  meets  $C_T$  by Lemma 1, so  $C_{\sigma_1} \cap C_T = D_1 \times E^1$  as in Lemma 2. This starts the recursion with  $S_1 = T \cup \{\sigma_1\}$ . Now suppose  $\{\sigma_1, \dots, \sigma_k\} \subset S_k \subset \Sigma$  such that  $C_{S_k} = D_k \times E^k$ , where  $E^k$  is a flat totally geodesic  $k$ -dimensional subspace complete in  $\hat{M}$ , such that each  $d \times E^k$  is stable under every  $\sigma \in S_k$  and is spanned by the  $\sigma_i$ -invariant lines through  $d$ .  $\Sigma$  preserves the splitting and  $\{\sigma_1, \dots, \sigma_k\} \setminus E^k$  is compact, so  $\Sigma$  induces a properly discontinuous group  $\bar{\Sigma}$  of isometries of  $D_k$ . Let  $T_k$  denote the torsion subgroup of  $\bar{\Sigma}$ ; it consists of all elements with fixed points on  $D_k$ , so  $\Sigma_k = \{\sigma \in \Sigma \mid \sigma \text{ preserves some } d \times E^k\}$  is its inverse image in  $\Sigma$ . Let  $D'_k$  denote the minimal set of  $T_k$  on  $D_k$ . Then  $C_{\Sigma_k} = D'_k \times E^k$  as above, and no  $\sigma \notin \Sigma_k$  preserves any  $d' \times E^k$ . If  $\Sigma \neq \Sigma_k$  choose  $\sigma_{k+1} \in \Sigma - \Sigma_k$  and define  $S_{k+1} = \Sigma_k \cup \{\sigma_{k+1}\}$ .  $C_{\sigma_{k+1}}$  meets  $C_{\Sigma_k}$  by Lemma 1, and their intersection  $C_{S_{k+1}} = D \times L$  as in Lemma 2. As  $\sigma_{k+1}$  cannot preserve any  $d' \times E^k$  of  $C_{\Sigma_k}$  now  $C_{S_{k+1}} = D_{k+1} \times E^{k+1}$  with the properties corresponding to those of  $D_k \times E^k$ . As  $\dim M < \infty$  this recursion terminates. Theorem 1 is proved for the case where  $\Sigma$  is abelian.

Let  $A$  be the last nontrivial term in the derived series of  $\Sigma$ . We have just seen that  $C_A = \bigcap_{\alpha \in A} C_\alpha$  is of the form  $D \times E_A$  where  $E_A$  is a flat totally geodesic subspace complete in  $\hat{M}$  such that  $A$  has finite kernel on  $E_A$  and  $A \setminus E_A$  is compact. Note that  $C_A$  is  $\Sigma$ -stable and that

the elements of  $\Sigma$  preserve its splitting as  $D \times E_A$ . Let  $\bar{\Sigma}$  denote the group of transformations of  $D$  induced by  $\Sigma$ . As  $A \setminus E_A$  is compact, proper discontinuity of  $\Sigma$  on  $M$ , thus on  $C_A$ , forces proper discontinuity of  $\bar{\Sigma}$  on  $D$ . As  $A$  acts trivially on  $D$ ,  $\bar{\Sigma} = \Sigma/B$  where  $A \subset B$ , so the derived series of  $\bar{\Sigma}$  is shorter than that of  $\Sigma$ . By induction on the length of the derived series now  $D$  has a flat totally geodesic subspace  $F$  with properties (i), (ii) and (iii) of Theorem 1 relative to  $\bar{\Sigma}$ . Now  $E = F \times E_A$  has the required properties relative to  $\Sigma$ . Q.E.D.

The proof of Theorem 1 did not really use discontinuity. We drop that condition on  $\Sigma$  by modifying Theorem 1 as follows.

**THEOREM 1'.** *Let  $\Sigma$  be a solvable group of semisimple isometries of  $M$ . Let  $\text{cl}_M(\Sigma)$  denote the closure of  $\Sigma$  in the Lie group of all isometries of  $M$ . Then  $M$  has a flat totally geodesic  $\text{cl}_M(\Sigma)$ -stable subspace  $E$  such that*

- (i)  *$E$  is complete in  $\bar{M}$ ,*
- (ii)  *$\text{cl}_M(\Sigma)$  acts with compact kernel on  $E$ , and*
- (iii)  *$\text{cl}_M(\Sigma) \setminus E$  is compact.*

*In particular every element of  $\text{cl}_M(\Sigma)$  is a semisimple isometry of  $M$ .*

{ Modify the argument of Theorem 1 as follows. Omit all mention of discontinuity. For  $\Sigma$  abelian let  ${}^0T$  be the torsion subgroup and  $T = \text{cl}_M({}^0T) \cap \Sigma$ ; then  $\text{cl}_M({}^0T) = \text{cl}_M(T)$  and is compact, and  $C^0_T = C_{\text{ol}_M(T)} \neq \emptyset$ . In the recursion step for  $\Sigma$  abelian let  ${}^0T_k$  be the torsion subgroup of  $\bar{\Sigma}$  and  $T_k = \text{cl}_{D_k}({}^0T_k) \cap \bar{\Sigma}$ . Then the argument proves Theorem 1'. }

**3. Some consequences of Theorem 1.** The following is the case in which  $\Gamma$  is the fundamental group of  $\Gamma \setminus M$ . It includes the case in which  $\Gamma \setminus M$  is a manifold.

**COROLLARY 1.** *Suppose that  $\Gamma$  acts freely on  $M$  and that  $\Sigma$  is a solvable subgroup of  $\Gamma$ . In the notation of Theorem 1,  $\Sigma$  is a Bieberbach group acting freely on  $E$ , and  $\Sigma \setminus E$  is a compact euclidean space form.*

**PROOF.**  $\Sigma$  acts freely on  $M$ , hence is torsion free, so  $\Phi = \{1\}$ . Thus  $\Sigma$  is a torsion free crystallographic group on  $E$ , i.e. is a Bieberbach group. Q.E.D.

We specialize Corollary 1 to the motivating case of Theorem 1.

**COROLLARY 2.** *Let  $\bar{M}$  be a compact riemannian manifold of sectional curvature  $K \leq 0$ . Let  $\Sigma$  be a solvable subgroup of the fundamental group  $\pi_1(\bar{M})$ . Then  $\Sigma$  is a Bieberbach group of some rank  $k$ ,  $0 \leq k \leq \dim \bar{M}$ , and  $\bar{M}$  contains a closed totally geodesic  $k$ -dimensional euclidean space form. In particular,  $\pi_1(\bar{M})$  has a solvable subgroup of finite index if and only if  $\bar{M}$  is flat (i.e. is a compact euclidean space form).*

The following corollary is known [11, Theorem 2.1] in the case  $\Sigma$  is central in  $\Gamma$ , and is related to Theorem 2.

**COROLLARY 3.** *Assume  $\Gamma \backslash M$  compact and let  $\Sigma$  be a solvable normal subgroup of  $\Gamma$ . Then  $M = D \times E$  isometrically in  $\bar{M}$ , where  $E$  is the euclidean space corresponding to  $\Sigma$  in Theorem 1, and where  $\Gamma$  preserves this product decomposition of  $M$ .*

**PROOF.** In the notation of Theorem 1,  $\bar{\Sigma} = \Sigma / \Phi$  is a crystallographic group on  $E$ . Let  $\bar{A} = A / \Phi$  be its translation subgroup.  $\Phi$  is a characteristic subgroup of  $\Sigma$ , and  $\bar{A}$  is a characteristic subgroup of  $\bar{\Sigma}$ , so  $A$  is a normal subgroup of  $\Gamma$ . Conjugation by any  $\gamma \in \Gamma$  induces a norm-preserving automorphism of the lattice  $\bar{A}$ . If  $\alpha \in A$  now the  $\Gamma$ -conjugacy class of  $\alpha$  is finite, so  $\delta_\alpha$  is bounded. Thus [10]  $\delta_\alpha$  is constant, and if  $\alpha \neq 1$  then  $M = D_1 \times E^1$  where  $d \times E^1$  is the  $\alpha$ -invariant line through  $d \in D$ . The recursion technique of Theorem 1 now gives  $M = D \times E$  where  $E$  corresponds to  $A$ , thus also to  $\Sigma$ , in the statement of Theorem 1. As  $A$  is normal in  $\Gamma$  this decomposition is invariant under every element of  $\Gamma$ . Q.E.D.

**COROLLARY 4.** *Let  $\bar{M}$  be a compact riemannian manifold of sectional curvature  $K \leq 0$ . Let  $\Sigma$  be a solvable normal subgroup of  $\pi_1(\bar{M})$ . Then  $\bar{M}$  has parallel orthogonal foliations  $\mathfrak{D}$  and  $\mathfrak{E}$  such that  $\dim \mathfrak{D} + \dim \mathfrak{E} = \dim \bar{M}$  and the leaves of  $\mathfrak{E}$  are flat closed totally geodesic submanifolds of dimension equal to the rank of  $\Sigma$  as a Bieberbach group. In particular, if  $\pi_1(\bar{M})$  is solvable then  $\bar{M}$  is flat.*

**PROOF.** Let  $M$  be the universal riemannian covering manifold of  $\bar{M}$ , so  $M = D \times E$  as in Corollary 3. Let  $\mathfrak{D}$  (resp.  $\mathfrak{E}$ ) be the foliation of  $\bar{M}$  whose leaves are the images of the  $D \times e$  (resp. of the  $d \times E$ ). Q.E.D.

Our last corollary answers a question raised by S. T. Yau in the case where  $\Gamma$  is free abelian and  $M$  is assumed complete.

**COROLLARY 5.** *Assume that  $M$  has empty boundary, but do not assume that the elements of  $\Gamma$  are semisimple isometries of  $M$ . Suppose that  $\Gamma$  has a subgroup  $\Sigma$  of finite index that is isomorphic to a discrete uniform subgroup of a simply connected solvable Lie group  $S$  with  $\dim S = \dim M$ . Then  $M$  is flat and complete in  $\bar{M}$  and  $\Gamma \backslash M$  is compact (so  $\Gamma$  is a crystallographic group on  $M$ ), and  $S$  is a vector group.*

**PROOF.** Both  $M$  and  $S$  are acyclic, and  $\Sigma$  is torsion free because  $S$  is torsion free, so  $M \rightarrow \Sigma \backslash M$  and  $S \rightarrow \Sigma \backslash S$  are universal  $\Sigma$ -bundles. Thus  $\Sigma \backslash M$  and  $\Sigma \backslash S$  are homotopy equivalent. Let  $n$  be the smallest integer such that  $H^q(\Sigma \backslash S; Z_2) = 0$  for  $q > n$ ; then  $n = \dim \Sigma \backslash S$  because  $\Sigma \backslash S$  is compact, so  $H^n(\Sigma \backslash M; Z_2) \neq 0$ . If the  $n$ -manifold  $\Sigma \backslash M$  were noncom-

pact then [8, Theorem 6.4] we would have  $H^n(\Sigma \setminus M; Z_2) = 0$ . Now  $\Sigma \setminus M$  is compact, hence flat by Corollary 2, and the assertions on  $M$  and  $\Gamma \setminus M$  follow. Thus we may assume  $\Sigma$  free abelian and the assertion on  $S$  follows. Q.E.D.

**4. The splitting theorem.** The following result implies a rigidity theorem for metrics of nonnegative curvature on compact product manifolds. If  $M$  is a riemannian symmetric space of noncompact type, it can be obtained by considering the algebraic hulls of the  $\Gamma_i$ .

**THEOREM 2.** *Suppose that  $\Gamma \setminus M$  is compact,  $\partial M = \emptyset$  and  $\Gamma = \Gamma_1 \times \Gamma_2$ . Then there is a  $\Gamma$ -invariant isometric splitting  $M = M_1 \times M'$  where  $\Gamma_1(M_1) = M_1$  and  $\Gamma_1 \setminus M_1$  is compact. If  $\Gamma_1$  is centerless then  $\Gamma_2$  acts trivially on  $M_1$ . If  $\Gamma$  is centerless then there is a  $\Gamma$ -invariant isometric splitting  $M = M_1 \times M_2$  inducing an isometric splitting  $\Gamma \setminus M = (\Gamma_1 \setminus M_1) \times (\Gamma_2 \setminus M_2)$ .*

**PROOF.**  $\Gamma$  has center  $Z = Z_1 \times Z_2$  where  $Z_i$  is the center of  $\Gamma_i$ . Corollary 3 gives a  $\Gamma$ -invariant isometric splitting  $M = M^* \times E$  where  $E$  is euclidean and  $Z$  is a translation lattice on  $E$ . If  $Z \neq \{1\}$  we replace  $\Gamma$  by  $\Gamma^* = \Gamma/Z$  and  $M$  by  $M^* = M/E$ , lowering the dimension but retaining our hypothesis. Repeating this construction we eventually replace  $\Gamma$  by a centerless group. Now we may assume that  $\Gamma$  has trivial center.

If  $S \subset M$ , let  $C(S)$  denote its convex hull, intersection of all closed convex subsets of  $M$  containing  $S$ .

Choose a compact set  $D_0 \subset M$  with  $\Gamma \cdot D_0 = M$ , and consider the closed convex set  $D = C(\Gamma_1 \cdot D_0) \cap C(\Gamma_2 \cdot D_0)$ . We prove that  $D$  is compact. If  $\gamma_1 \in \Gamma_1$  then  $\delta_{\gamma_1}$  is concave and bounded on  $\Gamma_2 \cdot D_0$ , thus bounded on  $C(\Gamma_2 \cdot D_0)$ . If  $\gamma_2 \in \Gamma_2$ , similarly  $\delta_{\gamma_2}$  is bounded on  $C(\Gamma_1 \cdot D_0)$ . If  $\gamma \in \Gamma$  now  $\delta_\gamma$  is bounded on  $D$ . If  $D$  is noncompact it has a sequence  $\{p_i\} \rightarrow \infty$ , and  $\Gamma$  has distinct elements  $\alpha_i$ , with  $\alpha_i(p_i) \in D_0$ . If  $\sigma \in \Gamma$  the displacements  $\delta_{\alpha_i \sigma \alpha_i^{-1}}(\alpha_i p_i) = \delta_\sigma(p_i)$  are bounded, so  $\{\alpha_i \sigma \alpha_i^{-1}\}$  is finite because  $D_0$  is compact. Passing to a subsequence now  $\alpha_i \sigma \alpha_i^{-1}$  is independent of  $i$ . Apply the argument in order to the elements of a finite generating set for  $\Gamma$ . Now the  $\alpha_i \alpha_i^{-1}$  are central and distinct in  $\Gamma$ . As  $\Gamma$  has trivial center,  $D$  is proved compact.

$\mathcal{Q}_0$  is the class of all closed convex  $\Gamma_1$ -invariant sets  $\emptyset \neq N \subset \Gamma_1 \cdot D (= C(\Gamma_1 \cdot D_0))$ . It is partially ordered by inclusion and its elements meet the compact set  $D$ , so  $\mathcal{Q}_0$  has a minimal element  $N_0$ .  $\mathcal{Q}$  is the class of all closed convex  $\Gamma_1$ -invariant sets  $N \neq \emptyset$  in  $M$  where  $\dim N = \dim N_0$ ,  $\Gamma_1 \setminus N$  is compact, and  $p \in N$  implies  $N = C(\Gamma_1 p)$ ;  $N_0 \in \mathcal{Q}$ . If  $N, N' \in \mathcal{Q}$  then  $p \mapsto \rho(p, N')$  assumes its minimum  $a \geq 0$  on  $N$ , say at  $p_0$ .  $N = C(\Gamma_1 p_0)$  and the arguments of Lemma 1 imply  $\rho(p, N') = a$  for all

$p \in N$ . Now  $C(N \cup N')$  splits isometrically in  $M$  as  $N \times I$  where  $p \times I$  is the minimal geodesic segment from  $p$  to  $N'$ . In particular, distinct sets in  $\mathcal{Q}$  are disjoint.

As  $\Gamma \backslash M$  is compact every  $C(\Gamma(p)) = M$ . Thus  $C(\Gamma_2 \cdot N_0) = M$ . If  $q \in M$  now there is a unique element  $N \in \mathcal{Q}$  through  $q$ .

Let  $N \in \mathcal{Q}$ . If  $\bar{p} \in \partial N$  choose  $p$  in the interior  ${}^0N$  and let  $J$  be the geodesic ray from  $p$  through  $\bar{p}$ . Let  $q \in J$  beyond  $\bar{p}$ , so  $q \in N$ . Let  $N' \in \mathcal{Q}$  through  $q$ . Since  $C(N \cup N') = N \times I$ , the triangle with vertices  $p$ ,  $q$  and  $r = (p \times I) \cap N'$  has right angles at  $p$  and  $r$ , which is impossible. Thus  $\partial N = \emptyset$ . As  $N$  is convex,  ${}^0N$  is totally geodesic in  $M$ . Thus  $N$  is totally geodesic without boundary in  $M$ .

Now  $\mathcal{Q}$  is a foliation of  $M$  by closed totally geodesic submanifolds; and  $N, N' \in \mathcal{Q}$  implies  $C(N \cup N') = N \times I$  is an isometric splitting as above. Thus the tangent spaces of the elements of  $\mathcal{Q}$  define a smooth  $\Gamma$ -invariant parallel distribution on  $M$ . With  $M_1 = N_0$  now  $M = M_1 \times M'$ ,  $\Gamma$ -invariant isometric splitting.  $\Gamma_1(M_1) = M_1$ ,  $\Gamma_1 \backslash M_1$  is compact and  $\Gamma_1$  acts trivially on  $M'$ .

Similarly  $M = M'' \times M_2$ ,  $\Gamma$ -invariant isometric splitting, such that  $\Gamma_2(M_2) = M_2$ ,  $\Gamma_2 \backslash M_2$  is compact and  $\Gamma_2$  acts trivially on  $M''$ . Note  $M = M_1 \times M_2$ , affine splitting.

Under  $M = M_1 \times M'$ ,  $\Gamma_2$  acts on  $M_1$  as a group  $\Sigma$  of isometries that centralize  $\Gamma_1$ . Thus every  $\sigma \in \Sigma$  is of bounded displacement on  $M_1$ , hence [10] is a translation along the euclidean factor. Let  $1 \neq \sigma \in \Sigma$ . Then  $M_1 = L \times N$ ,  $\Gamma_1$ -invariant isometric splitting, where  $\sigma$  translates the lines  $L \times x$ ,  $x \in N$ .  $\bar{M}_1 = \Gamma_1 \backslash M_1$  is fibred by the closures  $T_x$  of the images of the  $L \times x$ . The  $T_x$  are flat totally geodesic tori whose tangent spaces form a parallel distribution on  $\bar{M}_1$ . Choose a nontrivial circle group  $\bar{\tau} = \{\bar{\tau}_t\}$  of isometries of  $\bar{M}_1$  generated by a parallel vector field tangent to the  $T_x$ , such that  $\bar{\tau}_1 = 1$  and  $\bar{\tau}_t \neq 1$  for  $0 < t < 1$ . Then  $\bar{\tau}$  lifts to a 1-parameter group  $\tau = \{\tau_t\}$  of isometries of  $M_1$  centralized by  $\Gamma_1$ , and  $1 \neq \tau_1 \in \Gamma_1$ . That contradicts triviality of the center of  $\Gamma_1$ . Thus  $\Sigma = \{1\}$  and so  $\Gamma_2$  acts trivially on  $M_1$ . Similarly  $\Gamma_1$  acts trivially on  $M_2$ .

Under the  $\Gamma$ -invariant isometric splittings  $M'' \times M_2 = M = M_1 \times M'$ ,  $\Gamma_2$  acts trivially both on  $M''$  and on  $M_1$ . Thus  $M_1 = M''$ . Similarly  $M_2 = M'$ . Now the affine splitting  $M = M_1 \times M_2$  is isometric. The theorem follows. Q.E.D.

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