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A NONSTANDARD REPRESENTATION OF MEASURABLE SPACES AND L_∞

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The results given in this note were obtained by applying to measure theory the methods of nonstandard analysis developed by Abraham Robinson [5]. Amplifications of these results with proofs will be published elsewhere.[2] It is shown here that there are linear mappings from an arbitrary, real L_∞ space and its dual L_∞^* into Euclidean ω-space E^ω, where ω is an infinite integer. Finite valued, finitely additive measures on the underlying measurable space are also mapped onto elements of E^ω, and integrals are infinitesimally close to the corresponding inner products in E^ω. Yosida and Hewitt's representation of L_∞^* [6] is an immediate consequence of these results.

In general, we use Robinson's notation [5]. If we have an enlargement of a structure that contains the set R of real numbers, then $^*R$ denotes the set of nonstandard real numbers and $^*N$, the set of nonstandard natural numbers. A set S is called *finite if there is an internal bijection from an initial segment of $^*N$ onto S; a *finite set has all of the "formal" properties of a finite set. Given b and c in $^*R$, we write $b \simeq c$ if $b-c$ is in the monad of 0; when b is finite, we write $^\circ b$ for the unique, standard real number in the monad of b.

1. The partition P and bounded measurable functions. Let X be an infinite set and $\mathfrak{M}$ an infinite σ-algebra of subsets of X. Fix an enlargement of a structure that contains X, $\mathfrak{M}$, and the extended real numbers. There is a *finite, $^*\mathfrak{M}$-measurable partition P of $^*X$ such that P is finer than any finite $\mathfrak{M}$-measurable partition of X. That is, $P \subset {}^*\mathfrak{M}$ has the following properties:

(i) There is an infinite integer $\omega_P \in {}^*N$ and an internal bijection from $I = \{i \in {}^*N: 1 \leqq i \leqq \omega_P\}$ onto P. Thus we may write $P = \{A_i : i \in I\}$.

(ii) If i and j are in I and $i \neq j$, then $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$.

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(iii) $^*X = \bigcup_{i\in I} A_i$.

(iv) For each $B\in\mathfrak{M}$, let $I_B = \{i\in I : A_i \subset {}^*B\}$. Then I_B is *finite, and $^*B = \bigcup_{i\in I_B} A_i$.

(v) Let M be the set of $\mathfrak{M}$-measurable functions on X, and MB, the set of bounded functions in M. For each $f\in MB$ and $i\in I$, $\sup_{x\in A_i} {}^*f(x) - \inf_{x\in A_i} {}^*f(x) \simeq 0$.

Given the partition P, we let E denote the set of all internal mappings from I into $^*R$. The set E has all of the "formal" properties of Euclidean n-space. We shall write x_i instead of $x(i)$ for $x\in E$ and $i\in I$, and we shall write $x\cong y$ if x, $y\in E$ and $x_i \simeq y_i$, $\forall i\in I$. Let c_P denote a fixed internal choice function defined on I with $c_P(i)\in A_i \in P$ for each $i\in I$. Let T denote the mapping from MB into E defined by setting $T(f)(i) = {}^*f(c_P(i))$ for each $f\in MB$ and $i\in I$.

PROPOSITION 1. *Given f, g in MB and α, β in R, $T(\alpha f+\beta g) = \alpha T(f) + \beta T(g)$ and $T(f)\not\cong T(g)$ if $f\neq g$.*

2. **Measures and integration.** Let $\Phi(X, \mathfrak{M})$, or simply Φ, denote the set of all finitely additive real-valued functions μ on $\mathfrak{M}$ such that $\sup_{B\in\mathfrak{M}} |\mu(B)| < +\infty$. Let U be the mapping of Φ into E defined by setting $U(\mu)(i) = {}^*\mu(A_i)$ for each $\mu\in\Phi$ and $i\in I$. Clearly, U preserves addition and multiplication by real numbers. Conversely, if $e\in E$ and both $\sum_{i\in I}(e_i \vee 0)$ and $\sum_{i\in I}(-e_i\vee 0)$ are finite in $^*R$, let $\varphi(e)$ be that element of Φ such that for each $B\in\mathfrak{M}$, $\varphi(e)(B) = {}^\circ\sum_{i\in I_B} e_i$. (Note that we are writing $\sum$ instead of $^*\sum$ for the extension of the summation operator.) For each $\mu\in\Phi$, $\varphi(U(\mu)) = \mu$, but in general, $U(\varphi(e))\not\cong e$. If μ and ν are in Φ, then $U(\mu)\wedge U(\nu) \cong U(\mu\wedge\nu)$, and ${}^\circ\sum_{i\in I} |U(\mu)(i)| = |\mu|(X)$.

Let Φ_c and Φ_p be, respectively, the set of countably additive and the set of purely finitely additive elements of Φ. Yosida and Hewitt's Theorem 1.19 [6] has the following extension:

THEOREM 1. *There is a set $K\in{}^*\mathfrak{M}$ such that for all $\mu\in\Phi_c$, $|{}^*\mu|(K) \simeq 0$ and for all $\nu\in\Phi_p$, $|{}^*\nu|({}^*X - K) = 0$.*

Without loss of generality, we assume that $K = \bigcup\{A_i \in P : A_i \subset K\}$. If $\mu = \mu_c + \mu_p$ is the decomposition of an element μ in $\Phi = \Phi_c \oplus \Phi_p$, then when $A_i \subset {}^*X - K$, $U(\mu)(i) = U(\mu_c)(i)$ and when $A_i\subset K$, $U(\mu)(i) \simeq U(\mu_p)(i)$. We next show that there is a "maximum" null set for each $\mu\in\Phi^+$, and we extend the Hahn decomposition theorem for countably additive signed measures.

THEOREM 2. *Let μ be an arbitrary, finitely additive signed measure on $(X, \mathfrak{M})$. Let*

$$A_+ = \bigcup\{A_i \in P: {}^*\mu(A_i) > 0\}, \quad A_- = \bigcup\{A_i \in P: {}^*\mu(A_i) < 0\},$$

and

$$A_0 = \bigcup\{A_i \in P: {}^*\mu(A_i) = 0\}.$$

Then ${}^*\mu(A_0) = 0$, *and for each* μ*-null set* $B \in \mathfrak{M}$, ${}^*B \subset A_0$. *If there exists a* μ*-positive set* B_+ *and a* μ*-negative set* B_- *in* $\mathfrak{M}$ *with* $X = B_+ \cup B_-$ *and* $B_+ \cap B_- = \varnothing$, *then* $A_+ \subset {}^*B_+$, $A_- \subset {}^*B_-$, *and each* $A_i \in P$ *is either a* ${}^*\mu$*-positive set or a* ${}^*\mu$*-negative set.*

If we apply Theorem 2 to Lebesgue measure on the real line, we see that every standard real number is in the null set A_0.

Let $\Phi_1 = \{\mu \in \Phi: \mu(X) = 1 \text{ and } \forall B \in \mathfrak{M},\ \mu(B) = 0 \text{ or } \mu(B) = 1\}$. For each $j \in I$, let $\delta^j \in E$ be defined by setting $\delta^j_i = 0$ if $i \neq j$ and $\delta^j_j = 1$.

THEOREM 3. *For each* $j \in I$, $\varphi(\delta^j) \in \Phi_1$, *and for each* $\mu \in \Phi_1$, $U(\mu) = \delta^j$ *for some* $j \in I$. *Moreover, if* $\{x\} \in \mathfrak{M}$ *for each standard point* $x \in X$, *then the following are equivalent statements*:

(i) *Given* $j \in I$, $\varphi(\delta^j) \in \Phi_p$ *iff* $A_j \neq \{x\}$ *for any standard point* $x \in X$.

(ii) *Every free* $\mathfrak{M}$*-measurable ultrafilter* $\mathfrak{F} \subset \mathfrak{M}$ *contains a chain* $B_1 \supset B_2 \supset \cdots$, *with* $\bigcap_{n=1}^{\infty} B_n = \varnothing$.

If μ is a nonnegative finitely additive measure on $(X, \mathfrak{M})$ and $f \geqq 0$ is μ-integrable on X, then for each $B \in \mathfrak{M}$,

$$\int_B f\, d\mu \simeq \sum_{i \in I_B} \left(\inf_{x \in A_i} {}^*f(x) \right) {}^*\mu(A_i).$$

We can relate integration on X to the inner product "$\cdot$" in E as follows:

THEOREM 4. *If* $f \in MB$ *and* $\mu \in \Phi$, *then for each* $B \in \mathfrak{M}$,

$$\int_B f\, d\mu = {}^{\circ}\sum_{i \in I_B} {}^*f(c_P(i)) {}^*\mu(A_i).$$

In particular, $\int_X f\, d\mu \simeq T(f) \cdot U(\mu)$.

In general, Theorem 4 is false for unbounded functions $f \in M$. One can, however, find for each $f \in M$ an $\omega \in {}^*N$ such that if ${}^*f_\omega = -\omega \vee {}^*f \wedge \omega$, then for each $i \in I$, $\sup_{x \in A_i} {}^*f_\omega(x) - \inf_{x \in A_i} {}^*f_\omega(x) \simeq 0$. If $\mu \in \Phi$ and f is μ-integrable, then

$$\int_X f\, d\mu \simeq \sum_{i \in E} {}^*f_\omega(c_P(i)) {}^*\mu(A_i).$$

3. **The space L_∞ and its conjugate space.** Let $\mathfrak{N}$ be a proper subfamily of $\mathfrak{M}$ such that $\mathfrak{N}$ is closed under the formation of countable

unions and every $\mathfrak{M}$-measurable subset of an element of $\mathfrak{N}$ is an element of $\mathfrak{N}$. For each $f\in M$, set

$$\|f\|_\infty = \inf\{\alpha \in R: \{x \in X: |f(x)| > \alpha\} \in \mathfrak{N}\},$$

and let $M_0=\{f\in M:\|f\|_\infty<+\infty\}$. We say that two functions f and g in M_0 are equivalent if $\|f-g\|_\infty=0$, and we let L_∞ denote the usual Banach space of equivalence classes in M_0 with norm $\|\cdot\|_\infty$.

Given $\mathfrak{N}$, let $I_0=\{i\in I:A_i\in{}^*\mathfrak{N}\}$. Clearly, if $B\in\mathfrak{N}$, $I_B\subset I_0$. For each $f\in M_0$, let $T_0(f)$ be that element of E such that $T_0(f)(i)={}^*f(c_P(i))$ for $i\in I-I_0$ and $T_0(f)(i)=0$ for $i\in I_0$. Given f and g in M_0, $T_0(f)\cong T_0(g)\Rightarrow\|f-g\|_\infty=0\Rightarrow T_0(f)=T_0(g)$. Moreover, $\|f\|_\infty\simeq\max_{i\in I}|T_0(f)(i)|$. We may, therefore, consider T_0 to be a mapping of L_∞ into E; this mapping preserves addition and multiplication by standard real numbers.

For each functional F in the dual space L_∞^* of L_∞, let $V(F)$ be the element of E such that for all $i\in I$, $V(F)(i)={}^*F(\chi_{A_i})$, and let $\mu_F=\phi(V(F))$. It is easy to see that $U(\mu_F)=V(F)$. Yosida and Hewitt's representation of L_∞^* ([6, p. 53]) now has the following form:

THEOREM 5. *Let Φ_0 be the normed vector space $\{\mu\in\Phi:\mu(B)=0, \forall B\in\mathfrak{N}\}$ with norm given by $\|\mu\|=|\mu|(X)$. For each $F\in L_\infty^*$, let $\Theta(F)=\mu_F$. Then Θ is an isometric isomorphism from the Banach space L_∞^* onto Φ_0, and for each $F\in L_\infty^*$ and $f\in L_\infty$ we have*

$$F(f) = \int_X f\, d\mu_F \simeq V(F)\cdot T_0(f).$$

COROLLARY. *A nonzero functional $F\in L_\infty^*$ is multiplicative iff $U(\mu_F)=\delta^j$ for some $j\in I-I_0$.*

Assume now that there is a nonnegative $\mu\in\Phi_c$ such that $\mathfrak{N}=\{B\in\mathfrak{M}:\mu(B)=0\}$. If $f\in L_\infty$ and $\nu\in\Phi_c$ has the value $\nu(B)=\int_B f\,d\mu$ for each $B\in\mathfrak{M}$, then for each $i\in I-I_0$, ${}^*f(c_P(i))\simeq{}^*\nu(A_1)/{}^*\mu(A_i)$. To apply this result to probability theory, assume that $\mu(X)=1$ and choose a σ-algebra $\mathfrak{M}_1\subset\mathfrak{M}$. There is a *finite, ${}^*\mathfrak{M}_1$-measurable partition P_1 of ${}^*X$ such that P_1 is finer than any standard, finite $\mathfrak{M}_1$-measurable partition of X and such that for each $C\in P_1$, $C=\bigcup\{A_i\in P:A_i\subset C\}$. If $Y\in MB$ and $E(Y,\mathfrak{M}_1)$ is the conditional expectation of Y with respect to $\mathfrak{M}_1$, then for each $C\in P_1$ with $\mu(C)\neq 0$ and for each $x\in C$,

$${}^*E(Y,\mathfrak{M}_1)(x)\simeq\left[\sum_{A_i\in P;A_i\subset C}{}^*Y(c_P(i)){}^*\mu(A_i)\right]/{}^*\mu(C).$$

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