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CHARACTERISTIC CLASSES FOR PL MICRO BUNDLES

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0. In this note we shall outline the results about the cohomology of BSPL, where BSPL is the classifying space of the stable oriented PL micro bundles. In this paper p is always an odd prime number. The detailed version of these results will appear in [**18**].

THEOREM I. *As a Hopf algebra over* Z_p,
(i) $H_*(\mathrm{BSPL}:Z_p)\cong Z_p[b_1, b_2, \cdots]\otimes Z_p[\sigma(x_I)]\otimes\Lambda(\sigma(x_J))$;
(ii) $\Delta(b_j) = \sum_{i=0}^{j} b_i\otimes b_{j-i}$, $b_0=1$, $\deg b_j=4j$;
(iii) $\sigma(x_I)$, $\sigma(x_J)$ *are primitive elements.*

THEOREM II. *As a Hopf algebra over* $Z[1/2]$,
(i) $H^*(\mathrm{BSPL}:Z[1/2])/\mathit{Torsions}=Z[1/2]\ [R_1, R_2, \cdots]$;
(ii) $\Delta R_j=\sum_{i=0}^{j} R_i\otimes R_j$, $R_0=1$, $\deg R_j=4j$;
(iii) *In* $H^*(\mathrm{BSPL}:Q)=Q[p_1, p_2, \cdots]$, R_j *are expressed as follows:*

$$R_j = 2^{a_j}(2^{2j-1} - 1)\ \mathrm{Num}(B_j/4j)p_j + \mathrm{dec} \quad \textit{for some } a_j\in Z.$$

Let MSPL denote the spectrum defined by the Thom complex of the universal PL micro bundles. Let $A=A_p$ denote the mod p Steenrod algebra. $\phi:A\to H^*(\mathrm{MSPL}:Z_p)$ is defined by $\phi(a)=a(u)$, where $u\in H^0(\mathrm{MSPL})$ is the Thom class. The following is the conjecture of Peterson [**12**].

THEOREM III. *The kernel of* ϕ *is* $A(Q_0, Q_1)$, *the left ideal generated by Milnor's elements* Q_0 *and* Q_1.

1. The method to prove Theorem I is to compute the Serre spectral sequences associated to the fiberings, SPL→SF→F/PL→BSPL →BSF. The structures of $H_*(\mathrm{SF}:Z_p)$ and $H_*(\mathrm{BSF}:Z_p)$ were determined in [**9**], [**16**] and [**17**]. The homotopy type of F/PL is the deep result of Sullivan [**15**]. The first step is to study the H space structure of F/PL and the inclusion map k:SF→F/PL. The main tool in this step is the result of Sullivan [**15**], and its extension that tells the existence of the KO_P^* theory fundamental Thom classes for oriented PL micro bundles, where KO_P^* is 4 graded cohomology theory ob-

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tained from localizing the ordinary KO* theory at all odd primes P, cf. Sullivan [**15**].

PROPOSITION 1-1. *For any oriented* PL *disk bundle* $\pi:E\to X$ *over a finite complex* X *of fiber dimension* m, *there exists the Thom class* $u(\pi)\in\mathrm{KO}^m(E,\partial E)_P$ *with the following properties*:

(i) *Functorial, i.e. for* $f: Y\to X$, $u(f^!\pi)=f^!u(\pi)$.

(ii) $\varphi_H^{-1}phu(\pi)=L(\pi)^{-1}\in H^*(X,Q)$, *where* φ_H *is the Thom isomorphism, and* $L(\pi)$ *is the L polynomial of Hirzebruch for* $\pi:E\to X$.

(iii) $u(\pi\oplus 1)=\sigma u(\pi)$ *for* $\sigma:\mathrm{KO}^m(E,\partial E)_P\to\mathrm{KO}^{m+1}((E,\partial E)\wedge S^1)_P$, *the suspension.*

(iv) *Multiplicative mod torsions, i.e.* $u(\pi_1\oplus\pi_2)=u(\pi_1)\cdot u(\pi_2)$ *mod torsion elements.*

Let BO be the classifying space of the real vector bundles. This is a H space defined by Whitney sum of bundles. Let $\mathrm{BO}\langle 8N\rangle$ be the space obtained from BO by killing the homotopy groups $\pi_i(\mathrm{BO})$, $i<8N$. Then by Bott periodicity $\Omega^{8N}\mathrm{BO}\langle 8N\rangle=\mathrm{BO}\times Z$, and BO and $\mathrm{BO}\times 0$ coincide as H spaces. On the other hand there are products, $\mu_{M,N}:\mathrm{BO}\langle 8M\rangle\times\mathrm{BO}\langle 8N\rangle\to\mathrm{BO}\langle 8(M+N)\rangle$, obtained by tensor products of bundles. And we obtain the product $\mu:\Omega^{2M}\mathrm{BO}\langle 8M\rangle\times\Omega^{8N}\langle 8N\rangle=(\mathrm{BO}\times Z)\times(\mathrm{BO}\times Z)\to\Omega^{8(M+N)}\mathrm{BO}\langle 8(M+N)\rangle=\mathrm{BO}\times Z$. Restricting μ to the 1-component, we obtain a H space $\mu_\otimes:(\mathrm{BO}\times 1)\times(\mathrm{BO}\times 1)\to\mathrm{BO}\times 1$, and we denote this H space by $(\mathrm{BO}_\otimes,\mu_\otimes)$. Then there is the natural homotopy equivalence $i:\mathrm{BO}=\mathrm{BO}\times 0\to\mathrm{BO}\times 1=\mathrm{BO}_\otimes$. Let BO_P and $\mathrm{BO}_{\otimes P}$ be the spaces obtained by localizing BO and $\mathrm{BO}_\otimes$ at all odd primes P. And C_P denotes the class of abelian groups consisting of 2-torsion groups. Then Sullivan [**15**] defined the C_P homotopy equivalence

$$\sigma:F/\mathrm{PL}\to\mathrm{BO}_P$$

which is characterized by the formula,

$$\sigma^{**}(ph_1+ph_2+\cdots)=\tfrac{1}{8}(L_1+L_2+\cdots)\in H^{**}(\mathrm{F/PL},Q).$$

We define the map $\bar\sigma:\mathrm{F/PL}\to\mathrm{BO}_{\otimes P}$ by

$$\bar\sigma:\mathrm{F/PL}\xrightarrow{\sigma}\mathrm{BO}_P\xrightarrow{\times 8}\mathrm{BO}_P\xrightarrow{i_P}\mathrm{BO}_{\otimes P}.$$

PROPOSITION 1-2. *The* C_P *homotopy equivalence* $\bar\sigma$ *is a* H *space map.*

Let $f_N:S^{8N}\to\mathrm{BO}\langle 8N\rangle$ be the representative of the canonical generator $\pi_{8N}(\mathrm{BO}\langle 8N\rangle)\cong Z$. Then we obtain the map $g:\Omega^{8N}S^{8N}\to\mathrm{BO}\times Z$, and passing to the limit, $g:QS^0\to\mathrm{BO}\times Z$, where $QS^0=\lim\Omega^{8N}S^{8N}$. The 1-component Q_1S^0 of QS^0 becomes a H space by reduced join product and this H space is equivalent to SF. So we obtain a H map $g_1:\mathrm{SF}\to\mathrm{BO}_\otimes$.

PROPOSITION 1-3. *The maps* $g_1:\mathrm{SF}\to\mathrm{BO}_\otimes\to\mathrm{BO}_{\otimes P}$, *and* $\bar{\sigma}\circ k:\mathrm{SF}\to\mathrm{F/PL}\to\mathrm{BO}_{\otimes P}$ *coincide up to homotopy.*

COROLLARY 1-4. *As a Hopf algebra over* Z_p, $H_*(\mathrm{F/PL}:Z_p)=Z_p[a_1, a_2, \cdots]$. $\Delta a_j=\sum_{i=0}^{j} a_i\otimes a_{j-i}$, $a_0=1$. $\deg a_j=4j$.

Then Theorem I is obtained by tediously long calculations using Proposition 1-3, and results of [**17**] about $H_*(\mathrm{SF})$ and $H_*(\mathrm{BSF})$.

2. The first step to prove Theorem II is to compute the Bockstein spectral sequence, with $E_*^1=H_*(\mathrm{BSPL}:Z_p)$ and $E_*^\infty=(H_*(\mathrm{BSPL}:Z)/(\mathrm{Torsions}))\otimes Z_p$. And then studying the map

$$(H_*(\mathrm{F/PL}:Z)/\mathrm{Torsion})\otimes Z_p = H_*(\mathrm{F/PL}:Z_p)\to(H_*(\mathrm{BSPL}:Z)/\mathrm{Torsions})\otimes Z_p,$$

and

$$(H_*(\mathrm{BSO}:Z)/\mathrm{Torsions})\otimes Z_p\to(H_*(\mathrm{BSPL}:Z)/\mathrm{Torsions})\otimes Z_p,$$

we obtain Theorem II.

3. The essential part to prove Theorem III is the following proposition.

PROPOSITION 3-1. *There exists a oriented* PL *disk bundle* $\pi:E\to X$ *for some* X, *with the following properties*:

$Q_j(u)\neq 0$ *for* $j\geqq 2$, *where* $u\in H^*(E, \partial E:Z_p)$ *is the Thom class and* Q_j *are the Milnor elements.*

The construction of $\pi:E\to X$ is the following. Let K be a CW complex of the form,

$$K=S^{pr-1}\cup_p e^{pr}\cup_{\alpha_1} e^{(p+1)r}\cup_p e^{(p+1)r+1},\qquad r=2(p-1),$$

and let $f:K\to\mathrm{BSPL}$ be the map which represents β_1 in $j\circ f\circ i:S^{pr-1}\to K\to\mathrm{BSPL}\to\mathrm{BSF}$. Then f is represented by a PL disk bundle $\pi_f:E_f\to K$ of fiber dimension N, $N\gg 0$. Let π_p be the cyclic group of order p, and $W(\pi_p)=W$ be the free π_p acyclic complex. Then $P(\pi_f):W\times_{\pi_p}(E_f)^p\to W\times_{\pi_p}(K)^p$ is a oriented PL disk bundle of fiber dim pN. This is the bundle we seek.

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